RESOLVING EXTENSIONS OF FINITELY PRESENTED SYSTEMS

TODD FISHER

ABSTRACT. In this paper we extend certain central results of zero dimensional systems to higher dimensions. The first main result shows that if (Y, f) is a finitely presented system, then there exists a Smale space (X, F) and a u-resolving factor map $\pi_+ : X \to Y$. If the finitely presented system is transitive, then we show there is a canonical minimal u-resolving Smale space extension. Additionally, we show that any finite-to-one factor map between transitive finitely presented systems lifts through u-resolving maps to an s-resolving map.

1. INTRODUCTION

One cornerstone of the study of dynamical systems is the theory of hyperbolic dynamics introduced by Smale and Anosov in the 1960s. For compact spaces the property of hyperbolicity, in general, produces highly nontrivial and interesting dynamics. The best understood hyperbolic sets are those that are locally maximal (or isolated). For many years it was asked if every hyperbolic set can be contained in a locally maximal hyperbolic set: in [8] it is shown that this is not the case for any manifold with dimension greater than one.

This paper is in part an investigation into hyperbolic sets that are not locally maximal. If a hyperbolic set is not locally maximal one of the next properties one considers is the existence of a Markov partition, which is roughly a decomposition of the hyperbolic set into dynamically defined rectangles. In [8] it is shown that any hyperbolic set can be extended to one with a Markov partition.

The essential topological structure of a locally maximal hyperbolic set is captured with the notion of a Smale space, introduced by Ruelle [14] and simplified by Fried [9]. A **Smale space** is an expansive system with canonical coordinates (or a local product structure).

Date: November 2, 2007.
2000 *Mathematics Subject Classification.* 37C15, 37D05, 37B99.
Key words and phrases. Finitely presented, resolving, Smale space.

Fried defined a **finitely presented** dynamical system as an expansive system which is a factor of a shift of finite type. The finitely presented systems contain the Smale spaces and share a great deal of their structure; in particular, they are precisely the expansive systems that admit Markov partitions [9].

In this paper we extend certain central results of zero dimensional finitely presented systems (sofic shifts) to higher dimensional finitely presented systems. This investigation is motivated in part by hyperbolic sets that need not be locally maximal, and also as a generalization to higher dimension of the symbolic viewpoint. More specifically, the present work looks at resolving maps from Smale spaces to finitely presented systems; these are absolutely central to the zero-dimensional theory. A factor map from a space (X, f) to (Y, g) is **u-resolving** (**s-resolving**) if it is injective on unstable (stable) sets.

Every sofic shift has a canonical extension to a closely related SFT such that the factor map defining the extension is u-resolving. If the sofic shift is transitive, then the extension can be chosen to be transitive. Furthermore, in this setting there is a minimal extension in the class of u-resolving extensions of the sofic shift to transitive SFTs (i.e., every such extension factors through the minimal one). In dimension zero these covers permit the study of sofic shifts by the considerably more accessible SFTs. It is natural to seek such covers of finitely presented systems by Smale spaces in positive dimension.

Theorem 1.1. *If Y is a compact metric space and (Y, f) is finitely presented, then there exists a Smale space (X, F) and a u-resolving factor map $\pi^+ : X \to Y$. Furthermore, if (Y, f) is transitive, then there exists a transitive Smale space (X, F) and a u-resolving one-to-one almost everywhere map $\pi_+ : X \to Y$ (a residual set of points exists with a unique preimage).*

Our proofs use a (highly nontrivial) symbolic coding to construct the Smale space extensions. Fried gives a different proof of Theorem 1.1 in [9]. One aspect of this construction is that we are able to avoid a difficult part of Fried's argument [page 494, lines 28-29].

In general, there is no minimal extension to a Smale space for a finitely presented system, see [15, 16]. However, if (Y, f) is finitely presented and transitive we have the following result.

Theorem 1.2. *If (Y, f) is a transitive finitely presented system, then there exists a minimal extension among all transitive Smale spaces with u-resolving factor maps onto Y.*

Another important class of zero dimensional systems is almost of finite type (AFT). A subshift is **AFT** if it is the image of a transitive SFT via a factor map which is one-to-one on a nonempty open set. The AFT subshifts form the one naturally distinguished good class of sofic shifts. In particular, Boyle, Kitchens, and Marcus [4] show that a transitive AFT shift, has a canonical minimal extension to a transitive SFT where the extension is minimal in the class of *all* extensions to transitive SFTs. This extension is u-resolving, s-resolving and one-to-one almost everywhere.

We extend this result to positive dimensions. An expansive system is an **almost Smale space** if it is the image of a transitive Smale space via a factor map which is 1-1 on a nonempty open set.

Theorem 1.3. *Let (Y, f) be an almost Smale system, (X, g) be a transitive Smale space, and $\theta : X \to Y$ be 1-1 almost everywhere, u-resolving, and s-resolving. Let (X', g') be an irreducible Smale space and $\phi : X' \to Y$ a factor map. Then ϕ factors through θ.*

Approaching this subject with an eye to certain C^*-algebras Putnam [13] proved a surprising result: many factor maps lift through u-resolving maps to s-resolving maps. The next result extends Putnam's theorem to the case of finitely presented systems.

Theorem 1.4. *Let (X, f) and (Y, g) be transitive and finitely presented, and $\pi : X \to Y$ a finite-to-one factor map. Then there exist Smale spaces $(\tilde{X}, \tilde{f})$ and $(\tilde{Y}, \tilde{g})$ and factor maps ϕ, θ, and $\tilde{\pi}$, such that ϕ and θ are u-resolving, $\tilde{\pi}$ is s-resolving and the following diagram commutes:*

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\pi}"] \arrow[d, "\phi"'] & \tilde{Y} \arrow[d, "\theta"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

We remark that in the zero dimensional case, Boyle [3] obtained the above result for sofic systems, and even provided a meaningful generalization when the map is infinite-to-one. Boyle also establishes a canonical mapping property in dimension zero. From Theorem 1.4 we have the following corollary.

Corollary 1.5. *If (Y, f) is transitive and finitely presented, then there exists an SFT (Σ, σ), a Smale space (X, F), an s-resolving factor map $\alpha : \Sigma \to X$, and a u-resolving factor map $\beta : X \to Y$.*

The paper proceeds as follows: In Section 2 we provide some background results and definitions. In Section 3 we give preliminary results that will be used throughout the rest of the paper. These results give some basic facts on factor maps and resolving maps. In Section 4 we provide a proof of Theorem 1.1. In Section 5 we give a proof of Theorem 1.2. In Section 6 we give a proof of Theorem 1.3. Lastly, in Section 7 we provide a proof of Theorem 1.4 and give a meaningful extension of a magic word to the case of a finite-to-one factor map from an SFT to a Smale space.

Acknowledgement: The author would like to thank Mike Boyle for pointing out these problems and offering helpful advice and remarks.

2. Background

Before proceeding we review some useful concepts. Throughout the presented work we let X be a compact metric space and f be a homeomorphism of X. We now review expansive dynamics.

Definition 2.1. *Let X be a compact topological space with metric $d(\cdot,\cdot)$. A homeomorphism $f : X \to X$ is* **expansive** *if there exists a constant $c > 0$ such that for all $p, q \in X$, where $p \neq q$, there is an $n \in \mathbb{Z}$ with $d(f^n(p), f^n(q)) > c$.*

The constant $c > 0$ is called an **expansive constant**. The definition does not depend on the choice of metric, although the constant c may.

For $\epsilon > 0$ and $x \in X$ the **ϵ-stable set** is

$$W^s_\epsilon(x) = \{y \in X \mid \text{for all } n \geq 0, d(f^n(x), f^n(y)) < \epsilon\},$$

and the **ϵ-unstable set** is

$$W^u_\epsilon(x) = \{y \in X \mid \text{for all } n \geq 0, d(f^{-n}(x), f^{-n}(y)) < \epsilon\}.$$

For $x \in X$ and $f : X \to X$ an expansive homeomorphism the **stable set** is

$$W^s(x) = \{y \in X \mid \lim_{n\to\infty} d(f^n(x), f^n(y)) = 0\}$$

and the **unstable set** is

$$W^u(x) = \{y \in X \mid \lim_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) = 0\}.$$

In [9] it is shown that for any expansive system (X, f) there exists an **adapted metric** $d(\cdot,\cdot)$ and constants $\epsilon > 0$ and $\lambda \in (0, 1)$ such that

$$d(f(x), f(y)) \leq \lambda d(x, y) \text{ for all } y \in W^s_\epsilon(x) \text{ and}$$
$$d(f^{-1}(x), f^{-1}(z)) \leq \lambda d(x, z) \text{ for all } z \in W^u_\epsilon(x).$$

The following is a standard result for expansive systems.

Lemma 2.2. *If $f : X \to X$ is expansive with expansive constant c and $\epsilon < c/2$, then for any $x, y \in X$ the intersection $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ consists of at most one point.*

A system (X, f) has **canonical coordinates** if for all sufficiently small $\epsilon > 0$ there exists a $\delta > 0$ such that if $x, y \in X$ and $d(x, y) < \delta$, then $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ consists of one point.

A sequence $\{x_n\}_{j_1}^{j_2}$, where $-\infty \leq j_1 < j_2 \leq \infty$, is an **$\epsilon$-pseudo orbit** if $d(f(x_j), x_{j+1}) < \epsilon$ for all $j_1 \leq j < j_2$ if $j_1 > -\infty$ or for all $j < j_2$ if $j_1 = -\infty$. For a given $\delta, \epsilon > 0$ an ϵ-pseudo orbit is said to be **δ-shadowed** by $x \in X$ if $d(f^i(x), x_i) < \delta$ for all $j_1 \leq i \leq j_2$. A map f has the **pseudo orbit tracing property** (POTP) if for every $\delta > 0$ there is $\epsilon > 0$ such that every ϵ-pseudo orbit is δ-shadowed by a point $x \in X$. In [11] it is shown that a space (X, f) is a Smale space if and only if it has the pseudo-orbit tracing property.

A compact metric space X is **topologically transitive** for a map f if for any open sets U and V in X there is $n \in \mathbb{N}$ such that $f^{-n}(U) \cap V \neq \emptyset$. A space X is **topologically mixing** for f if for any nonempty open sets U and V in X there is $N \in \mathbb{N}$ such that $f^{-n}(U) \cap V \neq \emptyset$ for all $n \geq N$. A point $x \in X$ is a **chain recurrent** point if for all $\epsilon > 0$ there is an ϵ-pseudo orbit from x to x. The **chain recurrent set** denoted $R(f)$ consists of all chain recurrent points.

The following theorem is a useful description of the chain recurrent set of a Smale space. The theorem can be found in [1, p. 101-102], however, the terminology is quite different.

Theorem 2.3. *(Spectral Decomposition Theorem) Let (X, f) be a Smale space. Then there exists a finite collection of disjoint compact invariant sets B_i for $1 \leq i \leq l$ such that*

- *$f|_{B_i} : B_i \to B_i$ is transitive for all $1 \leq i \leq l$,*
- *$R(f) = \bigcup_{1 \leq i \leq l} B_i$, and*
- *$(B_i, f|_{B_i})$ is a Smale space.*

We now review some properties of shift spaces. Let $\mathcal{A}$ be a finite set of elements and $\mathcal{A}^{\mathbb{Z}}$ be the collection of all bi-infinite sequences of symbols from $\mathcal{A}$. A **block** (or **word**) over $\mathcal{A}$ is a finite sequence of symbols in $\mathcal{A}$. If $x \in \mathcal{A}^{\mathbb{Z}}$ and w is a block over $\mathcal{A}$, we say w occurs in x if there exist $i, j \in \mathbb{Z}$ such that $w = x[i, j]$. Let $\mathcal{F}$ be a collection of blocks over $\mathcal{A}$, called the **forbidden blocks**. Define $X_{\mathcal{F}}$ to be the subset of $\mathcal{A}^{\mathbb{Z}}$ which does not contain any block in $\mathcal{F}$.

A **shift space** (Σ, σ) consists of a set $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ such that $\Sigma = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden blocks in $\mathcal{A}$ and σ is the shift map, where $(\sigma s)_i = s_{i+1}$ for any $s \in \Sigma$.

Definition 2.4. *A* **subshift of finite type** *(SFT) is a shift space X having the form $X_{\mathcal{F}}$ for some finite set $\mathcal{F}$ of forbidden blocks.*

For a point $t \in \Sigma$ the **local unstable set of** s is denoted

$$W^u_{\mathrm{loc}}(t) = \{t' \in \Sigma \,|\, t_j = t'_j \text{ for all } j \leq 0\}$$

and the **local stable set of** s is denoted

$$W^s_{\mathrm{loc}}(t) = \{t' \in \Sigma \,|\, t_j = t'_j \text{ for all } j \geq 0\}.$$

For $t, t' \in \Sigma$ such that $t_0 = t'_0$ define the map

$$[t, t'] = \begin{cases} t_i & \text{for } i \geq 0 \\ t'_i & \text{for } i \leq 0 \end{cases}.$$

So $[t, t'] = W^s_{\mathrm{loc}}(t) \cap W^u_{\mathrm{loc}}(t')$.

A **factor map** from (X, F) to (Y, f) is a surjective semi-conjugacy. For (Y, f) a factor of an SFT (Σ, σ) by a factor map $\pi : \Sigma \to Y$ we define a relation $E_\pi \subset \Sigma \times \Sigma$ by $(s, t) \in E_\pi$ if $\pi(s) = \pi(t)$. It is clear that E_π is an equivalence relation. The following Lemma provides a useful characterization of E_π.

Lemma 2.5. [9] *Let (Σ, σ) be an SFT and (Y, f) a factor of (Σ, σ) such that $Y \simeq \Sigma / E_\pi$ for an equivalence relation E_π. Then (Y, f) is expansive if and only if E_π is an SFT.*

Let (Y, f) be expansive and $\epsilon < c/2$ for c an expansive constant of (Y, f). Following Fried in [9] we define

$$D_\epsilon = \{(x, y) \in Y \times Y \,|\, W^s_\epsilon(x) \text{ meets } W^u_\epsilon(y)\}$$

and $[\cdot, \cdot] : D_\epsilon \to Y$ so that $[x, y] = W^s_\epsilon(x) \cap W^u_\epsilon(y)$. It follows that $[\cdot, \cdot]$ is continuous.

Definition 2.6. *A* **rectangle** *is a closed set $R \subset Y$ such that $R \times R \subset D_\epsilon$.*

For R a rectangle and $x \in R$ denote

$$W^s(x, R) = R \cap W^s_\epsilon(x) \text{ and } W^u(x, R) = R \cap W^u_\epsilon(x).$$

Let $h_x : R \to W^s_\epsilon(x, R) \times W^u_\epsilon(x, R)$ so that $h_x(y) = ([x, y], [y, x])$. One easily checks that this is a homeomorphism with $h_x^{-1}(y, z) = [y, z]$. A rectangle R is **proper** if $R = \overline{\mathrm{int} R}$.

Definition 2.7. *Let (Y, f) be expansive with constant $c > 0$ and $0 < \epsilon < c/2$. A finite cover $\mathcal{R}$ of Y by proper rectangles with diameter$(R) < \epsilon$ for any $R \in \mathcal{R}$ is a* **Markov Partition** *if $R_i, R_j \in \mathcal{R}$, $x \in \text{int}R_i$, and $f(x) \in \text{int}R_j$, then*

- $f(W^s(x, R_i)) \subset R_j$ *and* $f^{-1}(W^u(f(x), R_j)) \subset R_i$, *and*
- $\text{int}(R_i) \cap \text{int}(R_j) = \emptyset$ *if* $i \neq j$.

Let (Y, f) be a finitely presented system with expansive constant c and assume there exists an SFT (Σ, σ) with a factor map $\pi : \Sigma \to Y$. Fix $\epsilon < c/2$. After passing to a higher block presentation we may assume that for each $j \in \mathcal{A}(\Sigma)$ the cylinder set $C_j = \{s \in \Sigma \,|\, s_0 = j\}$ has an image $R_j = \pi(C_j)$ that is a rectangle in Y. Furthermore, if $i, j \in \mathcal{A}(\Sigma)$ with an allowed transition from i to j, $y \in R_i$, and $f(y) \in R_j$, then

$$f(W^s(y, R_i)) \subset W^s(f(y), R_j) \text{ and } f^{-1}(W^u(f(y), R_j) \subset W^u(y, R_i).$$

This is called the **Markov property**.

For R_i and R_j rectangles the **unstable** j **boundary of** R_i is

$$\partial_u^j R_i = \{x \in R_i \,|\, x = \lim x_i, x_i \in W_\epsilon^s(x) \cap R_j - R_i\}.$$

Similarly, we can define the **stable** j **boundary of** R_i as

$$\partial_s^j R_i = \{x \in R_i \,|\, x = \lim x_i, x_i \in W_\epsilon^u(x) \cap R_j - R_i\}.$$

For a finitely presented system the **boundary of a rectangle** is

$$\partial R_i = \bigcup_j (\partial_u^j R_i \cup \partial_s^j R_i).$$

Let (Y, f) be a finitely presented system and (Σ, σ) an SFT extension of Y with a factor map π such that the image of each cylinder set for $i \in \mathcal{A}(\Sigma)$ is a rectangle R_i in Y. Following Fried in [9] we define the **symbol set**, **star**, and **second star** for a set $Y' \subset Y$ in the following manner:

$\mathcal{A}(Y') = \{i \in \mathcal{A}(\Sigma) \,|\, \text{there exists a point } x \in Y' \text{ where } x \in R_i\}$,
$\text{star}(Y') = \bigcup_{s \in \mathcal{A}(Y')} R_s,$ and
$\text{star}_2(Y') = \bigcup_{y \in \text{star}(Y')} \text{star}(y).$

3. Preliminary Results

In this section we provide some useful results concerning factor maps from Smale spaces to finitely presented systems. These results will be needed throughout the paper. The next lemma provides a criteria on periodic points for a factor map from a Smale space to a finitely presented system to be infinite-to-one.

Lemma 3.1. *Let (X, F) be a transitive Smale space, (Y, f) be finitely presented, and $\pi : X \to Y$ a factor map. If there exist convergent sequences $q_n \in Y$ and periodic points $p_n, p_n' \in X$ such that $\pi(p_n) = \pi(p_n') = q_n$, $p_n \neq p_n'$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} p_n = \lim_{n\to\infty} p_n'$, then π is infinite-to-one.*

Proof. The idea of the proof is to use the POTP to construct infinitely many points in X mapping to a point in Y. Let c_X and c_Y be expansive constants for X and Y, respectively. Fix $\gamma > 0$ such that $\gamma < c_X$ and if $x, x' \in X$ with $d(x, x') < \gamma$, then $d(\pi(x), \pi(x')) < c_Y/4$. We now use the pseudo-orbit tracing property of X and fix $\delta, \epsilon > 0$ such that $\max\{\epsilon, \delta\} < \gamma/4$ and any ϵ-pseudo orbit in X is δ-shadowed by an orbit in X. We know there exists an $N \in \mathbb{N}$ such that $d(p_N, p_N') < \epsilon$. Then there exists an $M \in \mathbb{N}$ such that

$$F^M(p_N) = p_N \text{ and } F^M(p_N') = p_N'.$$

Furthermore, since X is expansive there exists

$$0 < j < M \text{ such that } d(F^j(p_N), F^j(p_N')) > c_X.$$

Let s, s' denote the finite sequences

$$p_N, F(p_N), ..., F^{M-1}(p_N) \text{ and } p_N', F(p_N'), ..., F^{M-1}(p_N'),$$

respectively. Let ω, ω' be bi-infinite pseudo orbits that are concatenations of the segments s and s' where $\omega \neq \omega'$. Then ω and ω' are δ-shadowed by points x and x' in X, respectively. Then

$$\begin{aligned} d(F^j(x), F^j(x')) &\geq d(F^j(p_N, f^j(p_N')) - 2\delta \\ &> c_X - \gamma/2 > 0 \end{aligned}$$

and therefore $x \neq x'$. Furthermore, $\pi(x) = \pi(x') = q_N$ by the choice of γ. We know there are uncountably many such pseudo-orbits ω, and therefore $\#(\pi^{-1}(q_N)) = \infty$. □

We now use the previous lemma to prove a result about preimages of periodic points for finite-to-one maps between a Smale space and a finitely presented system.

Lemma 3.2. *If (X, F) is a transitive Smale space, (Y, f) is finitely presented, π is a finite-to-one factor map from X to Y, and $M = \min_{y\in Y} \#(\pi^{-1}(y))$, then there exists a dense open set $W \subset Y$ such that each periodic point in W has M preimages.*

Proof. Let $y \in Y$ such that $\#\pi^{-1}(y) = M$. Suppose there does not exist a neighborhood U of y such that each periodic point in U has exactly M preimages. Then there exists a sequence $q_n \to y$ such that each $q_n \in \mathrm{Per}(Y)$ has more than M preimages. Since π is continuous

and X compact we know that if $\{p_n\} \subset X$ is a sequence where $p_n \in \pi^{-1}(q_n)$ then any convergence subsequence of $\{p_n\}$ converges to a point of $\pi^{-1}(y)$.

Then there exists a point $x \in \pi^{-1}(y)$ and sequences $q_{n_j} \in Y$, $p_{n_j}, p'_{n_j} \in X$ such that

- q_{n_j} is a subsequence of q_n,
- $\pi(p_{n_j}) = \pi(p'_{n_j}) = q_{n_j}$,
- $p_{n_j} \neq p'_{n_j}$ for all $j \in \mathbb{N}$, and
- $\lim_{j\to\infty} p_{n_j} = \lim_{j\to\infty} p'_{n_j} = x$.

Lemma 3.1 then implies that π is infinite-to-one, a contradiction. Hence, there exists a neighborhood U of y such that each periodic point in U has exactly M preimages. Finally, the transitivity of X implies that Y is transitive and $W = \bigcup_{n\in\mathbb{Z}} f^n(U)$ is a dense open set in Y such that each periodic point in W has exactly M preimages. □

The next lemma shows that the bracket is preserved under π as long as the points in X are sufficiently close.

Lemma 3.3. *Let $\pi : X \to Y$ be a factor map from a Smale space (X, F) to a finitely presented system (Y, f). Then there exists a constant $\delta_\pi > 0$ such that if $x, x' \in X$ with $d(x, x') < \delta_\pi$, then $\pi([x, x']) = [\pi(x), \pi(x')]$.*

Proof. We assume that we are using adapted metrics on both Y and X. Let c be an expansive constant for Y and $\epsilon \in (0, c/2)$. So $W^s_\epsilon(y) \cap W^u_\epsilon(y')$ consists of at most one point for all $y, y' \in Y$. Let $\delta_\pi, \epsilon' > 0$ be sufficiently small such that we have the following:

- If $x, x' \in X$ and $d(x, x') < \epsilon'$, then $d(\pi(x), \pi(x')) < \epsilon$.
- There exists a δ_π such that if points $x, x' \in X$ and $d(x, x') < \delta_\pi$, then $W^u_{\epsilon'}(x) \cap W^s_{\epsilon'}(x')$ consists of one point in X.

Such a δ_π and ϵ' exist since X is a Smale space.

It then follows that if $d(x, x') < \delta_\pi$ that $[x, x']$ exists. We know that

$$d(F^{-n}([x, x']), F^{-n}(x)) < \epsilon' \text{ and } d(F^n([x, x']), F^n(x')) < \epsilon'$$

for all $n \in \mathbb{N}$. Since the map π is a factor map we have

$$\begin{aligned} &d(\pi F^{-n}([x, x']), \pi F^{-n}(x)) = d(f^{-n}(\pi[x, x']), f^{-n}\pi(x)) < \epsilon \text{ and} \\ &d(\pi F^n([x, x']), \pi F^n(x')) = d(f^n(\pi[x, x']), f^n\pi(x')) < \epsilon \text{ for all } n \in \mathbb{N}. \end{aligned}$$

This implies that

$$\pi([x, x']) \in W^u_\epsilon(\pi(x)) \cap W^s_\epsilon(\pi(x')) = [\pi(x), \pi(x')].$$

□

Claim 3.4. *Let (Y, f) be expansive with expansive constant $c > 0$ and $\epsilon < c/2$. If (Σ, σ) is an SFT extension of (Y, f) where the image of each cylinder set is a rectangle, then*

$$\pi(W^i_{\mathrm{loc}}(t)) = W^i(\pi(t), R_{t_0})$$

for each $t \in \Sigma$ and $i = u$ or s.

Proof. Let $\hat{t} \in W^u_{\mathrm{loc}}(t)$. So $\hat{t}_j = t_j$ for all $j \leq 0$. Hence,

$$\pi(\sigma^j \hat{t}) \in R_{t_j} \subset B_\epsilon(\pi(\sigma^j t))$$

for all $j \leq 0$. This implies that $\pi(\hat{t}) \in W^u_\epsilon(\pi(t)) \cap R_{t_0} = W^u(\pi(t), R_{t_0})$.

Now suppose that $y \in W^u(\pi(t), R_{t_0})$. So there exists $\hat{t}$ such that $\pi(\hat{t}) = y$ and $\hat{t}_0 = t_0$. Let $s = [t, \hat{t}]$. Then using a similar argument as in the previous lemma it is not hard to show that $\pi(s) = \pi([\hat{t}, t]) = [\pi(\hat{t}), \pi(t)] = y$ and $s \in W^u_{\mathrm{loc}}(t)$. □

4. Finitely presented systems have resolving extensions to Smale spaces

In this section we prove Theorem 1.1. The standard proof that every sofic shift has a u-resolving extension to an SFT uses the future cover of the sofic shift, see [10, p. 75]. The idea of the proof of Theorem 1.1 is very similar to the future cover construction. Namely, we will construct an SFT extension, related to the future cover, that codes the "different" unstable sets accumulating on a point. The SFT cover is zero dimensional, hence, we will need to factor the SFT cover appropriately to a Smale space such that there is a u-resolving factor map from the Smale space to the finitely presented system.

More specifically, the proof of Theorem 1.1 will proceed in the following steps: Let (Y, f) be a finitely presented system with a one-step SFT, (Σ, σ), an extension of (Y, f), such that the image of each cylinder set is a sufficiently small rectangle. In each rectangle we know there is a product structure, i.e., for a rectangle R if $x, y \in R$, then $[x, y]$ and $[y, x]$ is defined. Hence, the breakdown of a local product structure for a finitely presented system occurs at the boundaries of the rectangles. We use the rectangles then to detect where the product structure breaks down. We do this by coding the "different" unstable sets accumulating on a point, as given by the rectangles, and construct a one-step SFT, (Σ_+, σ), and a factor map π^0_+ from Σ_+ to Y. The coding given by the SFT, (Σ_+, σ), is too crude, in general, and so we define an equivalence relation $E_\alpha \subset \Sigma_+ \times \Sigma_+$ and a factor map $\alpha : \Sigma_+ \to X = \Sigma_+/E_\alpha$. The equivalence relation E_α is defined so that if, on a uniformly small scale, the unstable set detected by $\omega \in \Sigma_+$ and

$\omega' \in \Sigma_+$ is the same, then we say $(\omega, \omega') \in E_\alpha$. Finally, we define a factor map π_+ such that $\pi_+^0 = \pi_+\alpha$ and show that π_+ is u-resolving and X is a Smale space under the canonically induced map. So we will have the following diagram.

$$\begin{tikzcd}
& \Sigma_+ \arrow[dd, "\pi_+^0"] \arrow[dr, "\alpha"] & \\
\Sigma \arrow[dr, "\pi"'] & & X \arrow[dl, "\pi_+"] \\
& Y &
\end{tikzcd}$$

4.1. **Construction of Σ_+.** We now proceed with the construction. Let $d(\cdot, \cdot)$ be an adapted metric for Y under the action of f. We first need the rectangles in Y to be sufficiently small. Fix $c > 0$ an expansive constant for Y, and $0 < \epsilon < c/2$. Let (Σ, σ) be a one-step SFT extension of Y such that each cylinder set is a rectangle,

$$\max_{y \in Y}(\mathrm{diam}(\mathrm{star}_2(y))) < \epsilon/5 < c/10,$$

and for each $y \in Y \cap R_j$, where R_j is the image of the cylinder C_j under π, we have $f(W^u(y, R_j)) \subset W^u_\epsilon(y)$. We note that the existence of such a Σ is guaranteed by passing to a higher block presentation, if needed.

We now proceed with the construction of (Σ_+, σ) using (Σ, σ). For $s \in \Sigma$ define

$$E_i(s) = \{j \in \mathcal{A}(\Sigma) \mid W^u(\pi(\sigma^{-i}s), R_{s_i}) \cap R_j \neq \emptyset\}.$$

We will use the sets $E_i(s)$ to detect the different unstable sets accumulating on a point in Y.

Remark 4.1. *From the definition it follows that* $E_i(s) = E_{i-j}(\sigma^j(s))$ *for all* $s \in \Sigma$ *and all* $i, j \in \mathbb{Z}$.

We now define the one-step SFT, (Σ_+, σ), as an extension of (Y, f). Let (Σ_+, σ) be an SFT with symbol set

$$\mathcal{A}(\Sigma_+) = \{(v, j) \mid (E_i(s), s_i) = (v, j) \text{ for some } s \in \Sigma \text{ and } i \in \mathbb{Z}\}$$

and define allowable transitions from

$$(v, j) \in \mathcal{A}(\Sigma_+) \text{ to } (v', j') \in \mathcal{A}(\Sigma_+)$$

if there exists an $s \in \Sigma$ and $i \in \mathbb{Z}$ such that

$$(v, j) = (E_i(s), s_i) \text{ and } (v', j') = (E_{i+1}(s), s_{i+1}).$$

A point $\omega \in \Sigma_+$ is **canonically associated** with $s \in \Sigma$ if

$$\omega_i = (E_i(s), s_i) \text{ for all } i \in \mathbb{Z}.$$

For each $\omega \in \Sigma_+$ we know that if $\omega_j = (v_j, i_j)$, then i_j to i_{j+1} is an allowed transition in Σ for all $j \in \mathbb{Z}$. Furthermore, since (Σ, σ) is a one-step SFT we know that if $\omega \in \Sigma_+$ and $\omega_j = (v_j, i_j)$ for all $j \in \mathbb{Z}$, then there exists a unique $s \in S$ such that $s_j = i_j$ for all $j \in \mathbb{Z}$. Hence, there is a canonical map $\pi_+^0 : \Sigma_+ \to Y$ such that $\pi_+^0(\omega) = \pi(s)$. It is clear from the definition of Σ_+ that this is a well-defined map.

4.2. **Properties of Σ_+.** We now state and prove properties of Σ_+ that will be needed in the proof of Theorem 1.1. The next lemma shows that the set of canonically associated ω is dense in Σ_+.

Lemma 4.2. *Let $\omega \in \Sigma_+$, $k, l \in \mathbb{Z}$ with $k \leq l$. Then there exists a point $s \in \Sigma$ such that $(E_i(s), s_i) = \omega_i$ for all $k \leq i \leq l$.*

Proof. The proof proceeds by induction. From Remark 4.1 we may assume that $k = 0$. Fix $\omega \in \Sigma_+$. For $l = 0$ and $l = 1$ we know, by the definition of Σ_+, that there exists an $s \in \Sigma$ such that

$$(E_0(s), s_0) = \omega_0 \text{ and } (E_1(s), s_1) = \omega_1.$$

Now suppose there exists an $s \in \Sigma$ such that

$$(E_i(s), s_i) = \omega_i \text{ for all } 0 \leq i \leq m$$

where $m \geq 1$. Fix such an s and fix $s' \in \Sigma$ such that

$$(E_m(s'), s'_m) = \omega_m \text{ and } (E_{m+1}(s'), s'_{m+1}) = \omega_{m+1}.$$

We now construct the desired point by taking the bracket of $\sigma^m s'$ and $\sigma^m s$. Let

$$s'' = \sigma^{-m}[\sigma^m s', \sigma^m s].$$

From Claim 3.4 we know that

$$\sigma^i s'' \in W^u(\pi(\sigma^i s), R_{s_i}) \text{ and } s''_i = s_i \text{ for all } i \leq m.$$

Hence,

$$(E_i(s''), s''_i) = (E_i(s), s_i) = \omega_i \text{ for all } 0 \leq i \leq m.$$

We now show that

$$(E_{m+1}(s''), s''_{m+1}) = \omega_{m+1}.$$

We know that $s''_{m+1} = s'_{m+1}$ so we need to prove that $E_{m+1}(s'') = E_{m+1}(s')$.

Fix $j \in E_{m+1}(s')$. We now show that $j \in E_{m+1}(s'')$. Let

$$y \in W^u(\pi(\sigma^{m+1} s'), R_{s'_{m+1}}) \cap R_j.$$

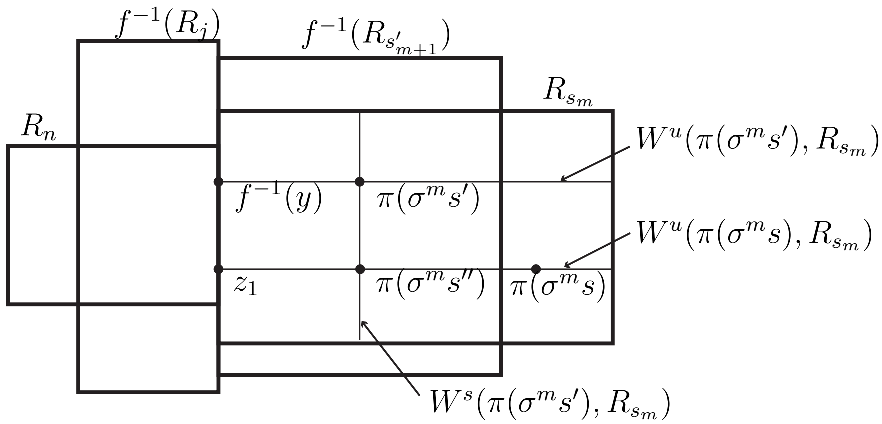

FIGURE 1. $j \in E_{m+1}(s'')$

We know from the Markov property that

$$f^{-1}(W^u(y, R_j)) \subset W^u(f^{-1}(y), R_n)$$

for some $n \in \mathcal{A}(\Sigma)$ where $n \to j$ is an allowed transition in Σ and

$$f^{-1}(W^u(\pi(\sigma^{m+1}s'), R_{s'_{m+1}})) \subset W^u(\pi(\sigma^m s'), R_{s'_m}).$$

Hence,

$$f^{-1}(y) \in W^u(\pi(\sigma^m s'), R_{s'_m}) \cap R_n$$

and $n \in E_m(s') = E_m(s) = E_m(s'')$.

Hence, there exists a point

$$z_0 \in W^u(\pi(\sigma^m s''), R_{s_m}) \cap R_n$$

and

$$z_1 = [f^{-1}(y), z_0] \in W^u(\pi(\sigma^m s), R_{s_m}) \cap R_n,$$

see Figure 1.

Since

$$z_1 \in W^s(f^{-1}(y), R_n) \subset f^{-1}(W^s(y, R_j))$$

and

$$z_1 \in W^s(f^{-1}(y), R_{s_m}) \subset f^{-1}(W^s(y, R_{s'_{m+1}}))$$

we know that

$$f(z_1) \in R_j \cap R_{s'_{m+1}}.$$

By construction we know that $f(z_1) \in W^u_\epsilon(\pi(\sigma^{m+1}s''))$. Therefore,

$$f(z_1) \in W^u(\pi(\sigma^{m+1}s''), R_{s'_{m+1}}) \cap R_j$$

and $j \in E_{m+1}(s'')$.

A similar argument shows that $E_{m+1}(s') \subset E_{m+1}(s'')$. Hence, $E_{m+1}(s'') = E_{m+1}(s')$. $\square$

Claim 4.3. *If $\omega \in \Sigma_+$, $\omega_0 = (v,i)$, and $j \in v$, then there exists a point $y \in W^u(\pi^0_+(\omega), R_i) \cap R_j$.*

Proof. The claim follows since canonically associated points are dense and the rectangles are closed. Let $s_k \in \Sigma$ such that $\pi(s_k) \to \pi^0_+(\omega)$ and $\omega_k \to \omega$ where ω_k is the sequence in Σ_+ canonically associated with s_k for all $k \in \mathbb{N}$. We may assume $(\omega_k)_o = (v,i)$ for all $k \in \mathbb{N}$. Then for each $k \in \mathbb{N}$ we know there exists

$$y_k \in W^u(\pi(s_k), R_i) \cap R_j.$$

By possibly taking a subsequence we may assume that the y_k are convergent to a point $y \in R_i \cap R_j$. Since R_i is a rectangle we know that $y \in W^u(\pi^0_+(\omega), R_i) \cap R_j$. $\square$

From the definition of Σ_+ and the previous claim we know that the if $(v,i) \in \mathcal{A}(\Sigma_+)$, then

$$\pi^0_+(C_{(v,i)}) = R_{(v,i)}$$

is a rectangle. Furthermore, $R_{(v,i)} \subset R_i$ and if $y \in R_i \cap R_{(v,i)}$, then

$$W^u(y, R_i) = W^u_\epsilon(y) \cap R_{(v,i)} = W^u(y, R_{(v,i)}).$$

To determine where the local product structure breaks down in Y we want to extend the rectangles $R_{(v,i)}$. To do this we define new rectangles, that we will denote $D_{(v,i)}$, that contain the sets $R_{(v,i)}$. Let $(v,i) \in \mathcal{A}(\Sigma_+)$ and define

$$D_{(v,i)} = \bigcup_{z \in R_{(v,i)}} (W^u_\epsilon(z) \cap (\bigcup_{j \in v} R_j)).$$

Claim 4.4. *For each $(v,i) \in \mathcal{A}(\Sigma_+)$ the set $D_{(v,i)}$ is a rectangle. Furthermore, the rectangles have the Markov property.*

Proof. The fact that the rectangles $D_{(v,i)}$ have the Markov property follows since the rectangles R_i have the Markov property. The set $D_{(v,i)}$ is a rectangle since canonically associated points are dense and the rectangles are closed. More specifically, we know that for $y \in D_{(v,i)}$ we have

$$\mathrm{diam}(D_{(v,i)}) \leq \mathrm{diam}(\mathrm{star}_2(y)) < \epsilon.$$

Fix $y_1, y_2 \in D_{(v,i)}$. Then there exists $z_1, z_2 \in R_{(v,i)}$ and $j, k \in v$ such that $y_1 \in W^u_\epsilon(z_1) \cap R_j$ and $y_2 \in W^u_\epsilon(z_2) \cap R_k$. From Claim 4.3 it follows

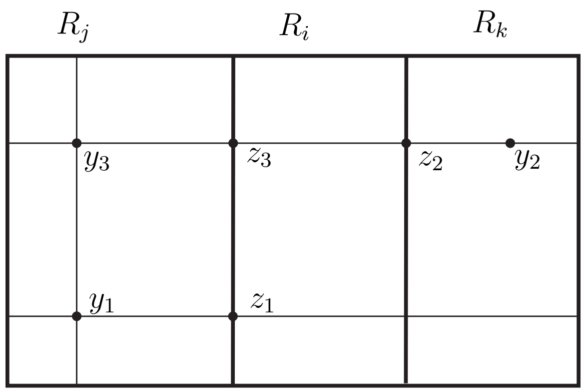

FIGURE 2. $D_{(v,i)}$ is a rectangle

that there exists a point $z_3 \in W^u(z_2, R_i) \cap R_j$. Then $[y_1, z_3] \in W^u_\epsilon(y_2)$, so $[y_1, z_3] = [y_1, y_2] = y_3$ exists and

$$y_3 \in W^u_\epsilon(z_2) \cap R_j$$

so $y_3 \in D_{(v,i)}$, see Figure 2. Similarly, we can show that $[y_2, y_1]$ exists and is contained in $D_{(v,i)}$. Hence, $D_{(v,i)}$ is a rectangle. □

4.3. **Construction of X.** We now show that the rectangles $D_{(v,i)}$ allow us to define a uniform constant $\delta > 0$ that can detect when two unstable sets coded by points in Σ_+ are different and when the unstable sets are simply coded differently. We will use this constant to define the equivalence relation, E_α, on points of Σ_+ such that $X = \Sigma_+/E_\alpha$.

Fix $\delta > 0$ sufficiently small so that $2\delta < c$ and such that the following hold:

(1) If $j, k \in \mathcal{A}(\Sigma)$ and $R_j \cap R_k = \emptyset$, then $d(R_j, R_k) > 2\delta$.
(2) If $y \in Y$ and $i \in \mathcal{A}(\Sigma)$ where $y \in R_i$, then $W^u_{2\delta}(y) \subset \mathrm{star}(W^u(y, R_i))$.

Property 2 says that if $s \in \Sigma$ and $y = \pi(s)$, then

$$W^u_{2\delta}(y) \subset W^u(y, D_{(E_0(s), s_0)}).$$

Define

$$B^u_\delta(\omega) = W^u_\delta(\pi^0_+(\omega)) \cap W^u(\pi^0_+(\omega), D_{\omega_0}).$$

Claim 4.5. *If $\omega \in \Sigma_+$, then $B^u_\delta(\omega) \subset f(B^u_\delta(\sigma^{-1}\omega))$.*

Proof. We know from Claim 4.4 that

$$f(W^u(\pi^0_+(\sigma^{-1}\omega), D_{\omega_{-1}}) \supset W^u(\pi^0_+(\omega), D_{\omega_0})$$

for all $\omega \in \Sigma_+$. Since Y has an adapted metric it also follows that

$$f(W^u_\delta(\pi^0_+(\sigma^{-1}\omega)) \supset W^u_\delta(\pi^0_+(\omega)).$$

Hence, $B^u_\delta(\omega) \subset f(B^u_\delta(\sigma^{-1}(\omega)))$. $\square$

4.3.1. *The relation E_α.* We now define the relation E_α on Σ_+.

Definition 4.6. *For $\omega, \omega' \in \Sigma_+$ we say $(\omega, \omega') \in E_\alpha$, or $\omega \sim_\alpha \omega'$, if $B^u_\delta(\sigma^j\omega) = B^u_\delta(\sigma^j\omega')$ for all $j \in \mathbb{Z}$.*

Since (Y, f) is expansive and $\delta < c/2$ we have

$$\pi^0_+(\omega) = \pi^0_+(\omega') \text{ for all } \omega \sim_\alpha \omega'.$$

Hence, the relation E_α is a subset of $E_{\pi^0_+}$. By definition the relation E_α is an equivalence relation. The idea is that if $(\omega, \omega') \in E_\alpha$, then ω and ω' represent the same unstable sets at $\pi^0_+(\omega)$ and are simply coded differently.

4.3.2. *The space X.* Let $X = \Sigma_+/E_\alpha$ and α be the canonical map from Σ_+ to X. Furthermore, let the canonically induced action on X be denoted by F. Define π_+ to be the factor map from X to Y such that $\pi^0_+ = \pi_+\alpha$. We will use the metric on X induced from the metric on Σ_+ and the equivalence relation E_α.

Remark 4.7. *If Y is a sofic shift, then the space X is the Fischer cover as described in* [6] *and* [7].

4.3.3. *The map π_+ is u-resolving.* The following definition and lemma will be useful in proving π_+ is u-resolving.

Definition 4.8. *An SFT E' a subset of an SFT E is* **forward closed** *if whenever $x \in E$ is backward asymptotic to E', then $x \in E'$.*
Lemma 4.9. [2] *If $\gamma = \beta\alpha$ is a factorization of factor maps, then β is u-resolving if and only if E_α is forward closed in E_γ.*

The next proposition will show that π_+ is u-resolving and X is finitely presented.

Proposition 4.10. *The relation E_α is closed, forward closed, and a 1-step SFT.*

Proof. To see that E_α is closed the idea is to take a sequence of points in E_α converging to some point in $\Sigma_+ \times \Sigma_+$. Since the unstable sets are all the same for the sequence this will also hold for the convergent point. Indeed, suppose there is a sequence $(\omega_k, \omega'_k) \in E_\alpha$ converging to a point $(\omega, \omega') \in \Sigma_+ \times \Sigma_+$. Since $E_{\pi^0_+}$ is closed it follows that $(\omega, \omega') \in E_{\pi^0_+}$ and $\pi^0_+(\omega) = \pi^0_+(\omega')$.

Fix a point $z \in B^u_\delta(\sigma^n\omega)$ for some $n \in \mathbb{Z}$. Note that if ω_k converges to ω that $B^u_\delta(\sigma^n\omega_k)$ converges to $B^u_\delta(\sigma^n\omega)$. It then follows that there

exists a sequence $z_k \in B^u_\delta(\sigma^n\omega_k)$ converging to z. Since $B^u_\delta(\sigma^n\omega_k) = B^u_\delta(\sigma^n\omega'_k)$ it follows that $z \in B^u_\delta(\sigma^n\omega')$. Hence,

$$B^u_\delta(\sigma^n\omega) = B^u_\delta(\sigma^n\omega') \text{ for all } n \in \mathbb{Z},$$

$(\omega,\omega') \in E_\alpha$, and E_α is closed.

We now show that the equivalence relation E_α is forward closed in $E_{\pi^0_+}$. The idea is the following: Suppose that $(\omega,\omega') \notin E_\alpha$ is backward asymptotic to E_α. Then the local unstable sets given by $B^u_\delta(\cdot)$ do not agree for some iterate of ω and ω'. Under backward iteration the point at which they do not agree will converge to the backward iterates of $\pi^0_+(\omega)$ and $\pi^0_+(\omega')$. Hence, any point in E_α for which (ω,ω') is backward asymptotic will have local unstable sets given by $B^u_\delta(\cdot)$ will also differ, a contradiction.

More precisely, let $(\omega,\omega') \in E_{\pi^0_+}$ be backward asymptotic to E_α. Suppose that $(\omega,\omega') \notin E_\alpha$. Then there is a $k \in \mathbb{Z}$ such that $B^u_\delta(\sigma^k\omega) \neq B^u_\delta(\sigma^k\omega')$. Since $B^u_\delta(\sigma^k\omega) \neq B^u_\delta(\sigma^k\omega')$ we may assume there exists a $z \in Y$ such that

$$z \in B^u_\delta(\sigma^k\omega) \text{ and } z \notin B^u_\delta(\sigma^k\omega').$$

From Claim 4.5 we know that

$$f^{-1}(B^u_\delta(\sigma^k\omega)) \subset B^u_\delta(\sigma^{k-1}\omega) \text{ and } f^{-1}(B^u_\delta(\sigma^k\omega')) \subset B^u_\delta(\sigma^{k-1}\omega').$$

This implies that

$$f^{-1}(z) \in B^u_\delta(\sigma^{k-1}\omega).$$

Furthermore, we know that

$$W^u_\delta(\pi^0_+(\sigma^k\omega')) \subset f(W^u_\delta(\pi^0_+(\sigma^{k-1}\omega'))) \text{ and}$$
$$W^u(\pi^0_+(\sigma^k\omega'), D_{\omega'_k}) \subset f(W^u(\pi^0_+(\sigma^{k-1}\omega'), D_{\omega'_{k-1}})).$$

This implies that

$$f^{-1}(z) \notin B^u_\delta(\sigma^{k-1}\omega').$$

Continuing inductively it follows that

$$f^{-n}(z) \in B^u_\delta(\sigma^{k-n}\omega) \text{ and } f^{-n}(z) \notin B^u_\delta(\sigma^{k-n}\omega') \text{ for all } n \in \mathbb{N}.$$

Since $z \in W^u_\epsilon(\pi^0_+(\sigma^k\omega))$ for all $n \in \mathbb{N}$ it follows that

$$d(f^{-n}(z), \pi^0_+(\sigma^{k-n}\omega)) \to 0 \text{ as } n \to \infty.$$

Fix $J \in \mathbb{N}$ such that $f^{-n}(z) \in B_{\delta/10}(\pi^0_+(\sigma^{k-n}\omega))$ for all $n \geq J$. Since (ω,ω') is backward asymptotic to E_α we know there exists a point $(\bar\omega,\bar\omega') \in E_\alpha$ and a subsequence of $\sigma^{-n}(\omega,\omega')$ converging to $(\bar\omega,\bar\omega')$. Denote $\bar\omega_0 = (\bar v, \bar i)$ and $\bar\omega'_0 = (\bar v', \bar i')$. Then for some $m \geq J$ sufficiently large we have

- $\omega_{k-m} = \bar\omega_0$,

- $\omega'_{k-m} = \bar{\omega}'_0$,
- $[\pi_+^0(\bar{\omega}), f^{-m}(z)] \in B^u_\delta(\bar{\omega})$, and
- $[\pi_+^0(\bar{\omega}'), f^{-m}(z)] \notin B^u_\delta(\bar{\omega}')$.

This follows since $D_{(\bar{v},\bar{i})}$ and $D_{(\bar{v}',\bar{i}')}$ are rectangles and contradicts the fact that $(\bar{\omega}, \bar{\omega}') \in E_\alpha$. Hence, $(\omega, \omega') \in E_\alpha$ and E_α is forward closed in $E_{\pi_+^0}$.

We now show the set E_α is a 1-step SFT. Let $(a,b), (c,d), (e,f) \in \mathcal{A}(\Sigma_+)$ such that there exist $(s,t), (s',t') \in E_\alpha$ where

$$\begin{array}{rcl} (a,b) & = & (s_{-1}, t_{-1}) \\ (c,d) & = & (s_0, t_0) = (s'_0, t'_0) \\ (e,f) & = & (s'_1, t'_1) \end{array}.$$

We now show that $[(s,t),(s',t')] \in E_\alpha$. Since $E_{\pi_+^0}$ is a 1-step SFT it follows that $[(s,t),(s',t')] \in E_{\pi_+^0}$. Also $[(s,t),(s',t')]$ is backward asymptotic to E_α so $[(s,t),(s',t')] \in E_\alpha$. Hence, $(a,b) \to (c,d) \to (e,f)$ is an allowed transition in E_α and E_α is a 1-step SFT. □

4.3.4. *Properties of* (X, F). Before proceeding to the proof of Theorem 1.1 we show some of the properties of the space (X, F).

Proposition 4.11. *If $X = \Sigma_+/E_\alpha$, then (X, F) is finitely presented and π_+ is u-resolving. Furthermore, if $s, s' \in \Sigma$ where $\pi(s) = \pi(s') = y$, then the points $\omega, \omega' \in \Sigma_+$ canonically associated with s and s', respectively, have $\alpha(\omega) = \alpha(\omega')$.*

The first part of the proof of the above proposition follows from the previous result and Lemma 2.5. The second part of the above proposition follows from the definition of E_α.

Remark 4.12. *Since the set of canonically associated points is dense in Σ_+ the above proposition implies that there is a canonical section $X' \subset X$ that is dense in X and a bijective map $\tau : Y \to X'$.*

We define rectangles in X as the image of the cylinder sets in Σ_+. For $(v,i) \in \mathcal{A}(\Sigma_+)$ we let

$$R_{(v,i)} = \{x \in X \mid \omega_0 = (v,i), \alpha(\omega) = x \text{ for some } \omega \in \Sigma_+\}$$

and if $x \in R_{(v,i)}$, then

$$W^u(x, R_{(v,i)}) = \{x' \in X \mid x' \in W^u(x), \pi_+(x') \in W^u(\pi_+(x), R_{(v,i)})\}.$$

The fact that these sets are rectangles follows directly from the definition.

The next lemma will be useful in showing that (X, F) is a Smale space. It will show that the rectangles $D_{(v,i)}$ are "reconnected" by the map α.

Lemma 4.13. *If $\pi_+(x) = \pi_+^0(\omega) = y$, $\alpha(\omega) = x$ and $\omega_0 = (v, i)$, then $\pi_+(W^u(x, R_{\omega_0})) = W^u(y, R_i)$. Furthermore, for each $m \in v$ there exists a $(v_m, m) \in \mathcal{A}(\Sigma_+)$ such that $W^u(x, R_{\omega_0}) \cap R_{(v_m,m)} \neq \emptyset$.*

Proof. The first statement follows since

$$\begin{aligned} W^u(y, R_i) &= W^u(\pi_+^0(\omega), R_{\omega_0}) \\ &= \pi_+(W^u(\alpha(\omega), R_{\omega_0})) \\ &= \pi_+(W^u(x, R_{\omega_0})). \end{aligned}$$

We now show the second statement. To do this we use points that are canonically associated. For canonically associated points we will see that the definition of E_α implies the result. Since the canonically associated points are dense we will see the result follows for all points in X. Indeed, let $s_k \in \Sigma$ be a sequence such that the sequence $\omega_k \in \Sigma_+$ canonical associated to s_k converges to ω. Fix $m \in v$ and a sequence $s'_k \in W^u_{\text{loc}}(s_k)$ such that for each $k \in \mathbb{N}$ the point $\pi(s'_k) \in R_i \cap R_m$. By reducing to a subsequence, if necessary, we may assume that the sequence s'_k is convergent to a point $s' \in \Sigma$ where $\pi(s') = y' \in W^u(y, R_i) \cap R_m$.

Let $\omega'_k \in \Sigma_+$ be the sequence canonically associated with s'_k. Then $\omega'_k \in W^u_{\text{loc}}(\omega_k)$ for all $k \in \mathbb{N}$ and ω'_k converges to some point $\omega' \in W^u_{\text{loc}}(\omega)$. Let $x' = \alpha(\omega')$. We want to show that there exists a $(v_m, m) \in \mathcal{A}(\Sigma_+)$ such that $x' \in R_{(v_m,m)}$.

Fix a sequence $s''_k \in \Sigma$ such that $(s''_k)_0 = m$ and $\pi(s''_k) = \pi(s'_k)$ for all $k \in \mathbb{N}$. Let $\omega''_k \in \Sigma_+$ be the sequence canonically associated with s''_k. Again by possibly taking a subsequence we may assume that s''_k converges to a point $s'' \in \Sigma$ and ω''_k converges to a point ω'' where $\pi(s'') = y' = \pi_+^0(\omega'')$. Furthermore, there exists a $(v_m, m) \in \mathcal{A}(\Sigma_+)$ such that $(\omega''_k)_0 = (v_m, m)$ for all $k \in \mathbb{N}$. From the definition of α we know that $(\omega''_k, \omega'_k) \in E_\alpha$ for all $k \in \mathbb{N}$. Hence, $(\omega'', \omega') \in E_\alpha$ and $x' \in R_{(v_m,m)} \cap W^u(x, R_{\omega_0})$. $\square$

Let $x \in X$ and $\omega \in \Sigma_+$ where $x = \alpha(\omega)$ and $\omega_0 = (v, i)$. For each $m \in v$ let $x_m \in W^u(x, R_{\omega_0}) \cap R_{(v_m,m)}$ and define

$$W^u(x, D_{\omega_0}) = \bigcup_{m \in v} W^u(x_m, R_{(v_m,m)}).$$

From the previous claim and lemma it is clear that $W^u(x, D_{\omega_0})$ is well defined and that the following proposition holds.

Proposition 4.14. *If $y = \pi_+(x) = \pi_+^0(\omega)$ and $\alpha(\omega) = x$, then*

$$W^u(y, D_{\omega_0}) = \pi_+(W^u(x, D_{\omega_0})).$$

Before proceeding to the proof of Theorem 1.1 we state a lemma that shows the space X distinguishes the "different" unstable sets accumulating on a point of Y.

Lemma 4.15. *If $x_1, x_2 \in X$ such that $\pi_+(x_1) = \pi_+(x_2) = y$, then $x_1 \neq x_2$ if and only if $\pi_+(W^u(x_1)) \neq \pi_+(W^u(x_2))$.*

Proof. Fix $\delta_0 > 0$ such that if $x, x' \in X$ and $d(x, x') < \delta_0$, then

$$d(\pi_+(x), \pi_+(x')) < \delta.$$

Suppose there exist points x_1 and x_2 such that $\pi_+(W^u(x_1)) \neq \pi_+(W^u(x_2))$. Since π_+ is injective on $W^u(x_1)$ and $W^u(x_2)$ we may suppose, by possibly taking a backward iterate of x_1 and x_2, that there exists

$$y' \in W^u_\delta(y) \cap \pi_+(W^u(x_1)) \text{ and } y' \notin \pi_+(W^u(x_2)).$$

Let $x_1' = \pi_+^{-1}(y') \cap W^u(x_1)$. By taking backwards iterates if needed we may assume that $x_1' \in W^u_{\delta_0}(x_1)$. Let $\omega_1, \omega_2 \in \Sigma_+$ such that $\alpha(\omega_1) = x_1$ and $\alpha(\omega_2) = x_2$. Then

$$B^u_\delta(\omega_1) \supset \pi_+(W^u_{\delta_0}(x_1)) \text{ and } B^u_\delta(\omega_1) \neq B^u_\delta(\omega_2)$$

since $y' \in B^u_\delta(\omega_1)$ and $y' \notin B^u_\delta(\omega_2)$. Hence, $x_1 \neq x_2$ since $(\omega_1, \omega_2) \notin E_\alpha$.

Now suppose that $x_1 \neq x_2$. Then by taking backwards iterates, if necessary, we may assume that there exist $\omega_1, \omega_2 \in \Sigma_+$ such that $\alpha(\omega_1) = x_1$, $\alpha(\omega_2) = x_2$, $B^u_\delta(\omega_1) \neq B^u_\delta(\omega_2)$, and there exists a point $y' \in W^u_\delta(y)$ such that

$$y' \in B^u_\delta(\omega_1) \text{ and } y' \notin B^u_\delta(\omega_2).$$

Hence, $\pi_+(W^u(x_1)) \neq \pi_+(W^u(x_2))$ since $\pi_+^{-1}(B^u_\delta(\omega_1))$ and $\pi_+^{-1}(B^u_\delta(\omega_2))$ are neighborhoods of x_1 in $W^u(x_1)$ and x_2 in $W^u(x_2)$, respectively. □

4.4. **Proof of Theorem 1.1.** Suppose (X, F) is not a Smale space. Then there exists some $\epsilon' > 0$ sufficiently small and sequences x_k and x_k' in X such that $d(x_k, x_k') < 1/k$ and $W^s_{\epsilon'}(x_k) \cap W^u_{\epsilon'}(x_k') = \emptyset$. By possibly taking a subsequence we may assume that x_k and x_k' converge to a point $x \in X$, and that $x_k \in R_{(v,i)}$ and $x_k' \in R_{(v',i')}$ for each $k \in \mathbb{N}$ where $(v, i), (v', i') \in \mathcal{A}(\Sigma_+)$.

By possibly taking a subsequence we may assume that the sequences $z_k = W^u_{\epsilon'/2}(x) \cap W^s_{\epsilon'/2}(x_k)$ and $z_k' = W^u_{\epsilon'/2}(x_k') \cap W^s_{\epsilon'/2}(x)$ satisfy the following, see Figure 3:

- the sequences are well defined,
- $W^u_{\epsilon'/2}(z_k') \cap W^s_{\epsilon'/2}(z_k) = \emptyset$ for all $k \in \mathbb{N}$,
- z_k converges to x, and
- z_k' converges to x.

Let ω and ω' be in Σ_+ such that $\alpha(\omega) = \alpha(\omega') = x$, $\omega_0 = (v, i)$, and $\omega_0' = (v', i')$. From the above it follows that for all $\epsilon'' > 0$ that

$$W^u(x, D_{\omega_0}) \cap B_{\epsilon''}(x) \neq W^u(x, D_{\omega_0'}) \cap B_{\epsilon''}(x).$$

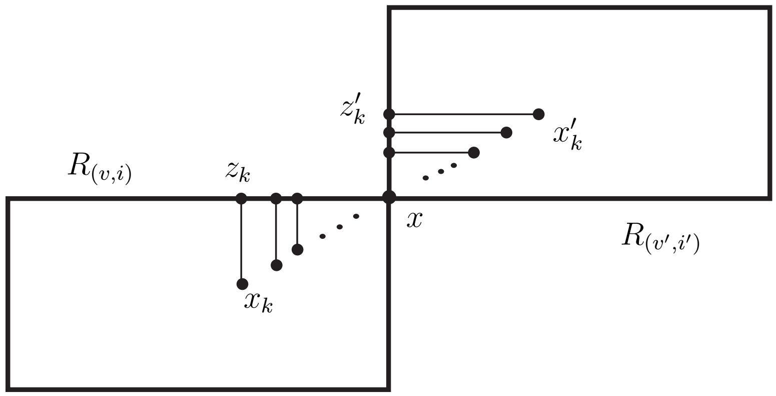

FIGURE 3. $D_{(v,i)}$ is a rectangle

The previous proposition implies that $B^u_\delta(\omega) \neq B^u_\delta(\omega')$, a contradiction. Therefore, (X, F) is a Smale space.

We now show that if (Y, f) is transitive, then (X, F) can be chosen to be transitive and π_+ a u-resolving one-to-one almost everywhere map. Since (X, F) is a Smale space there is a spectral decomposition such that $R(f)$ is decomposed into sets $\Lambda_1, ..., \Lambda_n$ such that $R(f) = \bigcup_{1\leq i\leq n} \Lambda_i$ each set Λ_i is closed, transitive, and has dense periodic points. Furthermore, we have

$$\mathrm{h}_{\mathrm{top}}(X) = \max_{1\leq i\leq n} \mathrm{h}_{\mathrm{top}}(\Lambda_i).$$

Therefore, X contains an irreducible component X_+ of maximal entropy. (For definitions and properties of topological entropy see [5, p. 36].) From the Spectral Decomposition Theorem we know that X_+ is a transitive Smale space. Furthermore, the image of X_+ under π_+ is Y since X_+ is of maximal entropy.

The next step is to show that π_+ is one-to-one on a residual set of Y. For each $(v, j) \in \mathcal{A}(\Sigma_+)$ the set $R_{(v,j)}$ forms a closed rectangle and the set of these rectangles forms a cover $\mathcal{M}$ covering Y. The cover $\mathcal{M}$ is not necessarily a Markov partition since the rectangles may not be proper. However, Fried in [9, p. 496-8] shows there is an open and dense set $\mathcal{D}$ of Y such that if $(v, j) \in \mathcal{A}(\Sigma_+)$ and $y \in \mathcal{D} \cap R_{(v,i)}$, then $y \in \mathrm{int}(R_{(v,i)})$. Let W be an open and dense set in Y such that each periodic point in W has the minimal number of preimages for π. If each point $p \in \mathrm{Per}(Y) \cap W$ has a unique pre-image under π^0_+, then it will follow that each point in $W \cap \mathcal{D}$ is contained in one rectangle $R_{(v,i)}$.

Hence, $V = \bigcap_{k\in\mathbb{Z}} f^k(\mathcal{D}\cap W)$ is a residual set in Y and each point in V would have a unique preimage in X under π_+. Hence, X_+ would be unique and each point in V would have a unique preimage under π_+ in X_+.

To complete the proof we show that each point $p \in \mathrm{Per}(Y)\cap W$ has a unique pre-image under π^0_+. Fix $p \in \mathrm{Per}(Y)\cap W$, let $s = \pi^{-1}(p)$, and $\omega \in \Sigma_+$ be the point canonically associated with s. Suppose $\omega' \in \Sigma_+$ and $\pi^0_+(\omega') = p$. We will show that $\omega'_0 = \omega_0$. Since p was arbitrary we then have that $f^n(p) = \pi(\sigma^n s)$ for all $n \in \mathbb{Z}$ and $\sigma^n\omega$ is canonically associated with $\sigma^n s$. Hence, $\omega'_n = \omega_n$ for all $n \in \mathbb{Z}$ and $\omega' = \omega$.

Let $\omega_0 = (v, i)$ and suppose there exists some $\omega' \in \Sigma_+$ such that $p = \pi^0_+(\omega')$ and $\omega'_0 = (v', i')$. Then $i = i'$ since p is contained in the interior of R_i. Fix a sequence $s^k \in \Sigma$ such that the canonically associated sequence $\omega^k \in \Sigma_+$ converges to ω'. We may assume the s^k are convergent, hence the sequence converges to s.

We now show that $v = v'$. Fix $j \in v'$. Then there exists a sequence $t^k \in \Sigma$ such that

$$\pi(t^k) \in W^u(\pi(s^k), R_i) \cap R_j.$$

We may assume that t^k converges to some point t. Hence,

$$\pi(t) \in W^u(\pi(s), R_i) \cap R_j$$

and $j \in v$.

Now fix $j \in v$. We will see that $j \in v'$ since p is a periodic point contained in the interior of R_i and the rectangles have the Markov property. Indeed, let

$$y \in W^u(p, R_i) \cap R_j$$

and $N \in \mathbb{N}$ be the period of p. Then $f^{-mN}(y) \in \mathrm{int}(R_i)$ for some $m \in \mathbb{N}$ since $p \in \mathrm{int}(R_i)$. Let t^k be a sequence of points in Σ such that the canonically associated points ω^k converge to ω' and $t^k_{-lN} = i$ for $0 \le l \le m$ and for all $k \in \mathbb{N}$. Let $y_k = [y, \pi(t^k)]$. From the Markov property of the rectangles we know that

$$f^{-mN}(y_k) = [f^{-mN}y, \pi(\sigma^{-mN}t^k)]$$

for all $k \in \mathbb{N}$. Furthermore, by the Markov property of the rectangles and the fact that the rectangles are proper we know that

$$\begin{array}{l} f^{-mN}(W^u(y, R_j)) \subset W^u(p, R_i) \text{ and} \\ W^s(f^{-mN}(y), R_i) \subset f^{-mN}(W^s(y, R_j)). \end{array}$$

Therefore, $f^{-mN}(y_k) \in f^{-mN}(W^s(y, R_j))$ and $y_k \in R_j$ for all $k \in \mathbb{N}$. Hence, $j \in v'$ and $v = v'$ and p has a unique preimage under π^0_+.

Let β be the restriction of π_+ to X_+. Since π_+ is u-resolving we know that β is u-resolving. Hence, (X_+, β) is a transitive Smale space

and β is a u-resolving one-to-one almost everywhere factor map from X_+ to Y. $\square$

5. Minimal resolving extensions for transitive finitely presented systems

In this section we prove Theorem 1.2. Specifically, we show that if (Y, f) is a transitive finitely presented system, then there exists a minimal u-resolving Smale space extension.

In general, a u-resolving factor map from a Smale space to a finitely presented system is injective, but not necessarily surjective, on unstable sets. However, this is not the case for resolving maps between Smale spaces.

Definition 5.1. *A factor map π from (X, F) to (Y, f) is* u**-bijective** *(*s**-bijective***) if for any $x \in X$ the map π is a bijection from $W^u(x)$ ($W^s(x)$) to $W^u(\pi(x))$ ($W^s(\pi(x))$).*

Theorem 5.2. [12] *If (X, F) is a transitive Smale space and (Y, f) is a Smale space, then a factor map π is u-resolving (s-resolving) if and only if π is u-bijective (s-bijective).*

We remark that it can be shown that if $\pi : X \to Y$ is a u-resolving factor map from a transitive Smale space (X, F) to a finitely presented system (Y, f), then there exists an open and dense set Win Y such that if $p \in \mathrm{per}(X)$ and $\pi(p) \in W$, then $\pi(W^u(p)) = W^u(\pi(p))$.

Proof of Theorem 1.2. Let (Y_+, f_+) be the transitive Smale space extension of (Y, f) constructed in the proof of Theorem 1.1 and π_+ the one-to-one almost everywhere factor map from Y_+ to Y. Let (X, g) be a transitive Smale space and $\alpha : X \to Y$ be a u-resolving factor map.

We will construct a map $\beta : X \to Y_+$ such that β is a u-resolving factor map.

$$\begin{tikzcd} X \arrow[rr, "\beta"] \arrow[dr, "\alpha"'] & & Y_+ \arrow[dl, "\pi_+"] \\ & Y & \end{tikzcd}$$

To construct the map β we will use an irreducible component of the fiber product.

We now form the fibered product of $\pi_+ : Y_+ \to Y$ and $\alpha : X \to Y$. Let

$$G = \{(x, z) \in X \times Y_+ \mid \alpha(x) = \pi_+(z)\},$$

the map $p_1 : G \to X$ be the coordinate projection onto X, and the map $p_2 : G \to Y_+$ be the coordinate projection onto Y_+. Endow G with the product metric. Then the map $h : G \to G$ defined as $h(x, z) = (g(x), f_+(z))$ is a homeomorphism. We then have the following commutative diagram.

$$\begin{tikzcd} & G \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "\alpha"'] & & Y_+ \arrow[dl, "\pi_+"] \\ & Y & \end{tikzcd}$$

It is not hard to see that (G, h) is a Smale space and p_1 and p_2 are u-resolving. Let G_+ be an irreducible component of maximal entropy. We denote the projection from G_+ to X as ρ_1 and the projection from G_+ to Y_+ as ρ_2. Since the maps π_+ and α are u-resolving it follows that the maps ρ_1 and ρ_2 are u-resolving. We will show that the map ρ_1 is injective, hence a homeomorphism. The map $\beta = \rho_2\rho_1^{-1}$ will then be a u-resolving factor map from X to Y_+.

We now proceed with the proof that ρ_1 is a homeomorphism. Since X, Y_+, and G_+ are transitive Smale spaces we know from Theorem 5.2 that $p_1(W^u(x, y)) = W^u(x)$ and $p_2(W^u(x, y)) = W^u(y)$ for any $(x, y) \in G_+$. Suppose there exist points $(x_1, y_1), (x_2, y_2) \in G_+$ with the same image under ρ_1. Then

$$x_1 = \rho_1(x_1, y_1) = \rho_1(x_2, y_2) = x_2.$$

Furthermore, we have

$$\begin{aligned} \pi_+(W^u(y_2)) &= \pi_+\rho_2(W^u(x_1, y_2)) = \alpha\rho_1(W^u(x_1, y_2)) \\ &= \alpha\rho_1(W^u(x_1, y_1)) = \pi_+\rho_2(W^u(x_1, y_1) \quad = \pi_+(W^u(y_1)). \end{aligned}$$

From Lemma 4.15 we know $y_1 = y_2$. Hence, ρ_1 is injective. □

6. Resolving extensions for almost Smale systems

The following result gives another characterization of almost Smale systems.

Proposition 6.1. *Let (X, F) be a transitive Smale space and (Y, f) be a finitely presented system. A factor map $\theta : X \to Y$ is 1-1 on an open set if and only if θ is u-resolving, s-resolving, and 1-1 on a residual set.*

Proof. Suppose θ is 1-1 on an open set. Then there exist open sets $U \subset X$ and $V \subset Y$ such that $\theta(U) = V$ and $\theta^{-1}(V) = U$. Since, f and

F are homeomorphisms it follows that for all $i \in \mathbb{Z}$ that $\theta(F^i(U)) = f^i(V)$ and $\theta^{-1}(f^i(V)) = F^i(U)$. Since (Y, f) is transitive it follows that there is an open dense set $\bigcup_{i\in\mathbb{Z}} f^i(V)$ where θ is 1-1.

Choose $\epsilon > 0$ and $\delta > 0$ such that for all points $x, x' \in X$ where $d(x, x') < \delta$ that $W^s_\epsilon(x) \cap W^u_\epsilon(x')$ consists of one point. Suppose θ is not u-resolving. Then there exist $x_1, x_2 \in X$ such that $x_1 \in W^u(x_2)$ and $\theta(x_1) = \theta(x_2)$. We may assume that $d(x_1, x_2) < \delta/2$. Let z be a transitive point for (X, F) such that the $d(z, x_1) < \delta/2$. It follows that the points $[x_1, z] = z_1$ and $[x_2, z] = z_2$ map to the same point under θ.

Let $x \in U$ and $r > 0$ such that $B_{4r}(x) \subset U$. Choose $n \in \mathbb{N}$ so that

- $d(F^{-n}(z), x) < r$,
- $d(F^{-n}(z), F^{-n}(z_1)) < r$, and
- $d(F^{-n}(z), F^{-n}(z_1)) < r$.

Then the points $F^{-n}(z_1), F^{-n}(z_2) \in U$ and $\theta(F^{-n}(z_1)) = \theta(F^{-n}(z_2))$, a contradiction. Hence, θ is u-resolving. Similarly, one can show θ is s-resolving.

Suppose θ is u-resolving, s-resolving, and 1-1 on a residual set. Let z be a transitive point in X and let $\epsilon > 0$ and $\delta > 0$ be given by the canonical coordinates on X. Define the rectangle

$$R_\delta(z) = \{x \in X \mid x = [z_1, z_2] \text{ for some } z_1 \in W^u_{\delta/2}(z) \text{ and } z_2 \in W^s_{\delta/2}(z)\}.$$

The rectangle $R_\delta(z)$ is a proper rectangle. Suppose that there exist points $x_1, x_2 \in \operatorname{int}(R_\delta(z))$ such that $\theta(x_1) = \theta(x_2)$. Then the points $[x_1, z], [x_2, z] \in W^u_{\delta/2}(z)$ get mapped to the same point under θ and θ is not u-resolving, a contradiction. Therefore, θ is 1-1 on $\operatorname{int}(R_\delta(z))$. $\square$

Proposition 6.2. *Let (X, f) and (Y, g) be transitive Smale spaces and $\pi : X \to Y$ be a u-resolving and s-resolving factor map. Then π is constant to one.*

Proof. The map π is bounded to one from a result in [2]. Let $y \in Y$ such that $\#(\pi^{-1}(y)) = k$ is maximal and fix $x \in \pi^{-1}(y)$. Fix $\epsilon > 0$ and $\delta > 0$ from the canonical coordinates on X. We may suppose that $\delta < \epsilon$.

We now show that π is 1-1 on $B_{\delta/2}(x)$. Let $x' \in X$ and $x_1, x_2 \in B_{\delta/2}(x')$ such that $\pi(x_1) = \pi(x_2)$. This implies that $\pi([x_1, x']) = \pi([x_2, x'])$. Since π is u-resolving we have $[x_1, x'] = [x_2, x']$, which implies that $x_1 \in W^s_\epsilon(x_2)$. Since π is s-resolving we then have $x_1 = x_2$ and π is 1-1 on $B_{\delta/2}(x)$.

Let $V = \bigcup_{x\in\pi^{-1}(b)} B_{\delta/2}(x)$. Then each point in V has exactly k pre-images for π. Since Y is transitive it follows that there is an open and dense set of points in Y with k pre-images. Hence any transitive point has k pre-images.

Fix $y \in Y$, a transitive point $z \in Y$, and sequence n_i such that $g^{n_i}(z)$ converges to y. Each $g^{n_i}(z)$ has k pre-images which we can order $z_i^1, ..., z_i^k$. By compactness of X we may suppose, by possibly taking a subsequence of $g^{n_i}(z)$, that each of the sequences z_i^j converge to a point in $\pi^{-1}(y)$. Since each $\delta/2$ neighborhood of a point in $\pi^{-1}(y)$ is 1-1, as shown in the previous paragraph, it follows that y has k pre-images. Hence, π is constant to one. $\Box$

Proof of Theorem 1.3. Let F be the fibered product for θ and ϕ. Denote ρ_1 and ρ_2 the canonical projections for F onto the first and second coordinates, respectively.

$$\begin{tikzcd} & F \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ X' \arrow[dr, "\phi"'] & & X \arrow[dl, "\theta"] \\ & Y & \end{tikzcd}$$

Since θ is u-resolving, s-resolving, and 1-1 on an open set it follows that ρ_1 is. From Proposition 6.2 it follows that ρ_1 is k to one for some constant $k \in \mathbb{N}$. Since θ is one to one on an open set it follows that $k = 1$. Therefore, ρ_1 is invertible. If $\rho = \rho_2\rho_1^{-1}$, then $\phi = \theta\rho$. The map ρ is clearly continuous, commuting, and onto. Hence, ρ is a factor map and ϕ factors through ρ. $\Box$

7. Lifting factor maps through u-resolving maps to s-resolving maps

We now show that every finite-to-one factor map between transitive finitely presented systems lifts through u-resolving maps to an s-resolving map between Smale spaces. The proof of Theorem 1.4 will proceed in two steps. First, we prove the following proposition.

Proposition 7.1. *Let (X, f) be a transitive finitely presented system, (Y, g) be a finitely presented system and π be a finite-to-one factor map from X to Y. Then there exist transitive Smale spaces $(\bar{X}, \bar{f})$ and $(\bar{Y}, \bar{g})$ and finite-to-one factor maps γ, β, and $\bar{\pi}$ such that the following diagram commutes.*

$$\begin{tikzcd} \bar{X} \arrow[r, "\bar{\pi}"] \arrow[d, "\gamma"'] & \bar{Y} \arrow[d, "\beta"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

Moreover, the maps γ and β are u-resolving.

Proof. Since X is transitive under f and π is a factor map we know that Y is transitive under g. Let (X', F) and $(\bar{Y}, \bar{g})$ be the minimal transitive Smale space extension of (X, f) and (Y, g), respectively, constructed in the proof of Theorem 1.1. Furthermore, let π_+ and β be the u-resolving almost one-to-one factor map from X' to X and $\bar{Y}$ to Y, respectively. The map $\pi' = \pi\pi_+$ is a finite-to-one factor map from X' to Y.

We now form the fibered product of $\bar{Y}$ and X' as in the last section. Let

$$G = \{(x, z) \in X' \times \bar{Y} \mid \pi(x) = \beta(z)\},$$

$p_1 : G \to X'$ the coordinate projection onto X, and $p_2 : G \to \bar{Y}$ the coordinate projection onto $\bar{Y}$.

$$\begin{tikzcd} G \arrow[r, "p_2"] \arrow[d, "p_1"'] & \bar{Y} \arrow[dd, "\beta"] \\ X' \arrow[d, "\pi_+"'] \arrow[dr, "\pi'"] & \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

We endow G with the product metric. Then the map $h : G \to G$ defined as $h(x, z) = (F(x), \bar{g}(z))$ is a homeomorphism.

As in the previous section it is not hard to see that (G, h) is a Smale space and p_1 is u-resolving. Let $\bar{X}$ be an irreducible component of maximal entropy, ψ be the restriction p_1 to $\bar{X}$, and $\bar{\pi}$ be the restriction of p_2 to $\bar{X}$. We then have the following commutative diagram.

$$\begin{tikzcd} \bar{X} \arrow[r, "\bar{\pi}"] \arrow[d, "\psi"'] & \bar{Y} \arrow[dd, "\beta"] \\ X' \arrow[d, "\pi_+"'] \arrow[dr, "\pi'"] & \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

Then $\gamma = \pi_+\psi$ is surjective and is a u-resolving factor map from $\bar{X}$ to X. The map $\bar{\pi}$ is surjective and a finite-to-one factor map. $\square$

Theorem 1.4 will follow from the above result and the next theorem.

Theorem 7.2. *Let $(\bar{X}, \bar{f})$ and $(\bar{Y}, \bar{g})$ be transitive Smale spaces and $\bar{\pi}$ a finite-to-one factor map from $\bar{X}$ to $\bar{Y}$. Then there exists transitive*

Smale spaces $(\tilde{X}, \tilde{f})$ and $(\tilde{Y}, \tilde{g})$ and finite-to-one factor maps $\tilde{\gamma}$, $\tilde{\beta}$, and $\tilde{\pi}$ such that the following diagram commutes.

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\pi}"] \arrow[d, "\tilde{\gamma}"'] & \tilde{Y} \arrow[d, "\tilde{\beta}"] \\ \bar{X} \arrow[r, "\bar{\pi}"'] & \bar{Y} \end{tikzcd}$$

Moreover, the maps $\tilde{\gamma}$ and $\tilde{\beta}$ are u-resolving, and $\tilde{\pi}$ is s-resolving.

Assuming the above result we now prove Theorem 1.4.

Proof of Theorem 1.4. From Theorem 7.2 and Proposition 7.1 we have the following commutative diagram.

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\pi}"] \arrow[d, "\tilde{\gamma}"'] & \tilde{Y} \arrow[d, "\tilde{\beta}"] \\ \bar{X} \arrow[r, "\bar{\pi}"] \arrow[d, "\gamma"'] & \bar{Y} \arrow[d, "\beta"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

Where $(\tilde{X}, \tilde{f})$, $(\tilde{Y}, \tilde{g})$, $(\bar{X}, F)$, and $(\bar{Y}, \bar{g})$ are transitive Smale spaces, and X and Y are transitive and finitely presented. The maps π, $\tilde{\pi}$, $\gamma\tilde{\gamma}$, and $\beta\tilde{\beta}$ are finite-to-one factor maps. Moreover, the map $\tilde{\pi}$ is s-resolving, and the maps $\gamma\tilde{\gamma}$ and $\beta\tilde{\beta}$ are u-resolving. $\Box$

The proof of Theorem 7.2 proceeds in a few steps. The first part is to build a u-resolving extension of $\bar{Y}$. We will not use the minimal extension, but instead construct a different u-resolving extension. The construction is similar to the construction in the proof of Theorem 1.1, however, we define a different relation E_θ that is a subset of E_α. We do this to help define the s-resolving map. The next step is to form an appropriate fiber product. In the last step we extend the notion of a magic word from symbolic dynamical systems to maps between finitely presented systems. This concept of a magic word is then used to construct the appropriate commuting diagrams.

Proof of Theorem 7.2. Let $c > 0$ be an expansive constant for both X and Y. Let $c/2 > \epsilon \geq \delta_0 > 0$ such that for any points $x, x' \in \bar{X}$ and $y, y' \in \bar{Y}$ where $d(x, x') < \delta_0$ and $d(y, y') < \delta_0$ that $W^s_\epsilon(x) \cap W^u_\epsilon(x')$ and $W^s_\epsilon(y) \cap W^u_\epsilon(y')$ consists of one point in $\bar{X}$ and $\bar{Y}$, respectively.

Let Σ be a transitive SFT such that there is a factor map $\nu : \Sigma \to \bar{X}$. Furthermore, we may assume that Σ generates a Markov partition on $\bar{X}$ and that the rectangles formed by the image of the cylinder sets in Σ are sufficiently small so that for any $x \in \bar{X}$ and $y \in \bar{Y}$ we have $\operatorname{diam}(\operatorname{star}_2(x)) < \delta_0/10$ and $\operatorname{diam}(\operatorname{star}_2(y)) < \delta_0/10$.

Define an SFT (Σ_+, σ) an extension of $\bar{Y}$ with factor map π^0_+, and $\delta > 0$ as in the proof of Theorem 1.1. For any $\omega \in \Sigma_+$ let $B^u_\delta(\sigma^j_+(\omega))$ be defined as in the proof of Theorem 1.1. We now define a relation E_θ on the points of Σ_+ such that we have the following commutative diagram.

$$\begin{tikzcd}
 & & \Sigma_+ \arrow[dd, "\pi^0_+"'] \arrow[dr, "\theta"] & \\
 & & & Y_+ \arrow[dl, "\pi_+"'] \\
\Sigma \arrow[r, "\nu"] & \bar{X} \arrow[r, "\bar{\pi}"] & \bar{Y} &
\end{tikzcd}$$

The map π_+ will be u-resolving and Y_+ will be a Smale space. However, in general, Y_+ will not be the minimal Smale space extension. More specifically, we construct Y_+ such that if $t, t' \in \Sigma$ where $t \neq t'$ satisfy $\bar{\pi}\nu(t) = \bar{\pi}\nu(t')$ and $t \in W^s(t')$, then $\theta(\omega) \neq \theta(\omega')$ where ω and ω' are in Σ_+ and are the canonically associated points of t and t', respectively.

Define

$$V^u_\delta(\sigma^j_+\omega) = \{k \in \mathcal{A}(\Sigma) \mid \omega_j = (v_j, i_j), k \in v_j, \text{ and } R_k \cap B^u_\delta(\sigma^j_+\omega) \neq \emptyset\}.$$

For any two points $\omega, \omega' \in \Sigma_+$ we say

$$(\omega, \omega') \in E_\theta \text{ if } V^u_\delta(\sigma^j_+\omega) = V^u_\delta(\sigma^j_+\omega) \text{ for all } j \in \mathbb{Z}.$$

By the definition of E_θ we see that E_θ is an equivalence relation. For points $\omega, \omega' \in \Sigma_+$ where $(\omega, \omega') \in E_\theta$ we notice that it is not enough simply that $B^u_\delta(\sigma^j_+\omega) = B^u_\delta(\sigma^j_+\omega')$ for all $j \in \mathbb{Z}$, we need the sets to be in the same rectangles in v_j and v'_j where $\omega_j = (v_j, j)$ and $\omega'_j = (v'_j, j')$ for all $j \in \mathbb{Z}$.

To show how this differs from the definition of α notice that if f_a is hyperbolic toral automorphism of $\mathbb{T}^2$ and Σ generates a sufficiently small Markov partition of $\mathbb{T}^2$ and Σ_+ and α are defined as in the proof of Theorem 1.1, then we know that the set Σ_+/E_α and the induced action on this space is the minimal u-resolving extension to a Smale space. Hence, $\Sigma_+/E_\alpha = \mathbb{T}^2$, the induced action is just f_a, and π_+ is the identity map. However, the space Σ_+/E_θ will not be $\mathbb{T}^2$. Specifically, Figure 4 shows a neighborhood of a point y in $\mathbb{T}^2$ and rectangles for a Markov partition for f_a. Then the point y will have at least three

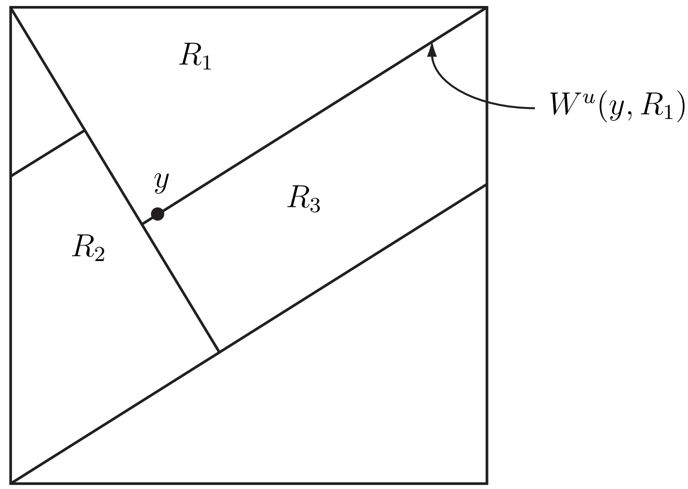

Figure 4. Preimages for θ

preimages in Σ_+/E_θ. Namely, one containing 1 and 2 in $V^u_\delta(\cdot)$ but not 3, one containing 1, 2, and 3 in $V^u_\delta(\cdot)$, and one containing 2 and 3 but not 1 in $V^u_\delta(\cdot)$.

Let $Y_+ = \Sigma_+/E_\theta$ and θ be the canonical factor map from Σ_+ to Y_+. Let π_+ be the canonical factor map so $\pi^0_+ = \pi_+\theta$. We will show that $E_\theta \subset E_\alpha$, the set Y_+ is a Smale space under the action of the induced map, and π_+ is u-resolving. The proof of these facts is very similar to the proof in Theorem 1.1 that Σ_+/E_α is a Smale space and the resulting factor map is u-resolving.

Lemma 7.3. *The map π_+ is u-resolving, and Y_+ is a Smale space.*

Proof. From the size of δ and $\mathrm{star}_2(y)$ for all $y \in \bar{Y}$ we know that if $(\omega, \tilde{\omega}) \in E_\theta$, then $\pi^0_+(\omega) = \pi^0_+(\tilde{\omega})$. Furthermore, if $V^u_\delta(\sigma^j_+\omega) = V^u_\delta(\sigma^j_+\tilde{\omega})$, then $B^u_\delta(\sigma^j_+\omega) = B^u_\delta(\sigma^j_+\tilde{\omega})$. It then follows that $E_\theta \subset E_\alpha$.

We first show E_θ is closed. Let $\omega^k, \tilde{\omega}^k$ be sequences in Σ_+ converging to ω and $\tilde{\omega}$, respectively, such that $(\omega^k, \tilde{\omega}^k) \in E_\theta$. Then $(\omega, \tilde{\omega}) \in E_\alpha$. We may assume that $\omega^k_0 = \omega_0$ and $\tilde{\omega}^k_0 = \tilde{\omega}_0$ for all k.

Suppose $(\omega, \tilde{\omega}) \notin E_\theta$. We may assume that $V^u_\delta(\omega) \neq V^u_\delta(\tilde{\omega})$ and there exists a

$$j \in V^u_\delta(\omega) \text{ such that } j \notin V^u_\delta(\tilde{\omega}).$$

Then for k sufficiently large we have $B^u_\delta(\omega_k) \cap R_j \neq \emptyset$ since

- $B^u_\delta(\omega_k)$ is open relative to $W^u(\pi^0_+(\omega_k), D_{(\omega_k)_0})$,
- R_j is closed, and
- the unstable sets vary continuously inside $D_{(\omega_k)_0}$.

Hence, $j \in V^u_\delta(\omega_k) = V^u_\delta(\tilde{\omega}_k)$. If $\tilde{\omega}_0 = (\tilde{v}, \tilde{i})$, then $j \in \tilde{v}$. Since $\pi^0_+(\omega) = \pi^0_+(\tilde{\omega})$ and $B^u_\delta(\omega) \cap R_j \neq \emptyset$ we have $j \in V^u_\delta(\tilde{\omega})$.

We now show that E_θ is forward closed. Let $(\omega, \omega') \in E_{\pi^0_+}$ be backward asymptotic to E_θ. Then $(\omega, \omega') \in E_\alpha$. Suppose for some $j \in \mathbb{Z}$ that $V^u_\delta(\sigma^j \omega) \neq V^u_\delta(\sigma^j \omega')$. If $V^u_\delta(\sigma^{j-1}\omega) = V^u_\delta(\sigma^{j-1}\omega')$, then using the Markov property of Σ we know that $V^u_\delta(\sigma^j\omega) = V^u_\delta(\sigma^j\omega')$. Hence, for all $n \in \mathbb{N}$ we have $V^u_\delta(\sigma^{j-n}\omega) \neq V^u_\delta(\sigma^{j-n}\omega')$. We may suppose there is a

$$l_0 \in V^u_\delta(\sigma^j\omega) \text{ such that } l_0 \notin V^u_\delta(\sigma^j\omega')$$

and $y_0 \in R_{l_0} \cap B^u_\delta(\sigma^j\omega)$. Then $y_1 = f^{-1}(y_0) \in R_{l_1}$ such that l_1 to l_0 is an allowed transition for Σ and $l_1 \in V^u_\delta(\sigma^{j-1}\omega)$. Furthermore, $l_1 \notin V^u_\delta(\sigma^{j-1}\omega')$ since otherwise l_0 would be in $V^u_\delta(\sigma^j\omega')$, and

$$d(\pi^0_+(\sigma^{j-1}\omega), y_1) \leq \lambda d(\pi^0_+(\sigma^j\omega), y_0)$$

where $\lambda \in (0,1)$ is a fixed constant depending on the adapted metric $d(\cdot,\cdot)$. Continuing inductively we see that for each $n \in \mathbb{N}$ there is a $l_n \in \mathcal{A}(\Sigma)$ such that

$$l_n \in V^u_\delta(\sigma^{j-n}\omega) \text{ and } l_n \notin V^u_\delta(\sigma^{j-n}\omega'),$$

and a point $y_n = f^{-n}(y_0)$ such that $y_n \in R_{l_n}$ and

$$d(\pi^0_+(\sigma^{j-n}\omega), y_n) \leq \lambda^n d(\pi^0_+(\sigma^j\omega), y_0).$$

Fix a subsequence $(\sigma_+^{-n_j}\omega, \sigma_+^{-n_j}\omega')$ converging to a point $(\bar{\omega}, \bar{\omega'}) \in E_\theta$. We may suppose that l_{n_j} are all the same. Since

$$y_{n_j} \to \pi^0_+(\bar{\omega}) = \pi^0_+(\bar{\omega'}) \text{ as } j \to \infty$$

we know that

$$l_{n_j} \in V^u_\delta(\bar{\omega}) \text{ and } l_{n_j} \notin V^u_\delta(\bar{\omega'}).$$

Indeed his shows $\pi^0_+(\bar{\omega}) \in R_{l_{n_j}}$ and if $\bar{\omega}_0 = (\bar{v}, \bar{i})$ and $\bar{\omega'}_0 = (\bar{v'}, \bar{i})$, then $l_{n_j} \in \bar{v}$ and $l_{n_j} \in \bar{v'}$, a contradiction. Therefore, $(\omega, \omega') \in E_\theta$ and E_θ is forward closed.

From Lemma 4.9 we know that π_+ is u-resolving and Y_+ is finitely presented. We now show that Y_+ is a Smale space. If $t, t' \in \Sigma$ such that $\bar{\pi}\nu(t) = \bar{\pi}\nu(t')$, and $\omega, \omega' \in \Sigma_+$ are canonically associated with t and t', respectively, then the definition of V^u_δ implies that $(\omega, \omega') \in E_\theta$. Then we can prove similar statements to Lemma 4.13 and Proposition 4.14 replacing θ with α. Hence, using the same argument as in the proof of Theorem 1.1 we can show that Y_+ is a Smale space. □

Now let $\tilde{Y}$ be a transitive component of maximal entropy in Y_+ from the Spectral Decomposition Theorem. Let $\tilde{\beta}$ be the restriction of π_+ to $\tilde{Y}$. As stated previously this implies that $\tilde{\beta} : \tilde{Y} \to \bar{Y}$ is a finite-to-one

u-resolving factor map. Let Σ'_+ be an irreducible component of Σ_+ such that $\theta(\Sigma'_+) = \tilde{Y}$.

Let G_0 be the fiber product of Σ and $\tilde{Y}$ and maps ρ_1 and ρ_2 the projections onto Σ and $\tilde{Y}$, respectively. Fix G a transitive component of G_0 of maximal entropy from the Spectral Decomposition Theorem and let π_1 and π_2 be the restriction of ρ_1 and ρ_2, respectively, to G.

$$
\begin{tikzcd}
 & & G \arrow[dll, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\Sigma \arrow[r, "\pi"] \arrow[drr, "\tilde{\pi}\pi"'] & \bar{X} \arrow[dr, "\bar{\pi}"] & & \tilde{Y} \arrow[dl, "\tilde{\beta}"'] \\
 & & \bar{Y} &
\end{tikzcd}
$$

We now construct a factor of G that will give us the desired commuting diagram. Let τ send a point $(s, y) \in G$ to $(\pi(s), y)$ and let $\tilde{X}$ be the image of G under τ. It is not hard to see that the space $\tilde{X}$ is a transitive Smale space since $\bar{X}$ is a Smale space and a factor of Σ by π. Furthermore, there exist maps $\tilde{\gamma}$ and $\tilde{\pi}$ projections from $\tilde{X}$ onto $\bar{X}$ and $\tilde{Y}$, respectively. We then have the following commutative diagram.

$$
\begin{tikzcd}
 & & G \arrow[ddll, "\pi_1"'] \arrow[d, "\tau"'] \arrow[ddr, "\pi_2"] & \\
 & & \tilde{X} \arrow[dl, "\tilde{\gamma}"] \arrow[dr, "\tilde{\pi}"'] & \\
\Sigma \arrow[r, "\pi"] \arrow[drr, "\tilde{\pi}\pi"'] & \bar{X} \arrow[dr, "\bar{\pi}"] & & \tilde{Y} \arrow[dl, "\tilde{\beta}"'] \\
 & & \bar{Y} &
\end{tikzcd}
$$

Lemma 7.4. *The map π_2 is s-resolving.*

We postpone the proof of the above lemma to the next subsection and proceed with the proof of the theorem. Since $\tilde{\beta}$ is u-resolving we know that π_1 and $\tilde{\gamma}$ are u-resolving. The maps τ and $\tilde{\pi}$ are s-resolving since π_2 is s-resolving and $\tilde{\pi}\tau = \pi_2$. From Theorem 5.2 we know for each $(t, y) \in G$ we have $\tau(W^s(t, y)) = W^s(\tau(t), y)$. Therefore, $\tilde{\pi}$ is s-resolving and we have the desired commutative diagram. $\square$

7.1. **Proof of Lemma 7.4.** Before proceeding to the proof of Lemma 7.4 we extend the notion of a magic word in symbolic dynamics to finite-to-one maps from SFT's to Smale spaces. For a definition and basic properties of a magic word for finite-to-one maps between shift spaces see [10, p. 303]. These arguments will show that each transitive point y in Y under f has a unique preimage for $\pi_+$and that $\bar{Y}$ is the unique irreducible component in Y_+ and the set G is transitive so $G = G_0$. From this we will be able to conclude that π_2 is s-resolving.

Claim 7.5. *Let Σ be a transitive one-step SFT and ϕ be a finite-to-one factor map from Σ to a transitive expansive system (Y, f). Then there exists $K \in \mathbb{N}$ so that if w, w', and w'' are words of size K, and if w is pairwise related to w' by π, and w' is pairwise related to w'' by π, then w is pairwise related to w'' by π.*

Proof. Suppose that no such K exists. Then for each $n \in \mathbb{N}$ sufficiently large there exists words w_n, w'_n, and w''_n where $|w_n| \geq 2n$ and such that w_n is pairwise related to w'_n, and w'_n is pairwise related to w''_n, and w_n is not pairwise related to w''_n. Then for each n there exists points $s^n, (s')^n, (s'')^n \in \Sigma$ such that $s^n[-n, n] \subset w_n$, $(s')^n[-n, n] \subset w'_n$, and $(s'')^n[-n, n] \subset w''_n$. Since Σ is compact we know that the sequence $(s^n, (s')^n, (s'')^n) \in \Sigma^3$ has a convergent subsequence $(s^{n_j}, (s')^{n_j}, (s'')^{n_j}) \in \Sigma^3$ converging to a point $(s, s', s'') \in \Sigma^3$. Since w_{n_j} and w'_{n_j} are pairwise related for all $j \in \mathbb{N}$ and w'_{n_j} and w''_{n_j} are pairwise related for all $j \in \mathbb{N}$ we know that $\phi(s) = \phi(s')$ and $\phi(s') = \phi(s'')$. So $\phi(s) = \phi(s'')$ and for all j sufficiently large we have w_{n_j} pairwise related to w''_{n_j}, a contradiction. Hence, such a $K \in \mathbb{N}$ exists. □

Fix a $K \in \mathbb{N}$ as in the previous claim. Let $m, n \in \mathbb{Z}$ where $m \leq -K$ and $K \leq n$. Let w be a word such that there exists an $s \in \Sigma$ where $s[m, n] = w$ and let $\mathcal{W}_{m,n}(w)$ be the maximal family of words pairwise related words by ϕ containing w. Let $m \leq j \leq n$ and

$$\deg(\mathcal{W}_{m,n}(w), j) = \#\{\bar{w}_j \,|\, \bar{w} \in \mathcal{W}_{m,n}(w)\}.$$

We define

$$\deg(\mathcal{W}_{m,n}(w)) = \min_{m \leq j \leq n} \deg(\mathcal{W}_{m,n}(w), j)$$

and $d = \min_{\mathcal{W}_{m,n}(w)} \deg(\mathcal{W}_{m,n})$. Let $D = \min\{\#\phi^{-1}(y) \,|\, y \in Y\}$. Then we know that $D \geq d$. We will show that $D = d$.

Fix $\mathcal{W}_{m,n}(w)$ such that $d = \deg(\mathcal{W}_{m,n}(w))$. By possibly decreasing m, increasing n, and shifting the jth coordinate of w we may assume that $\deg(\mathcal{W}_{m,n}(w)) = \deg(\mathcal{W}_{m,n}(w), 0)$. Then for words $\bar{w}, \bar{w}' \in \mathcal{W}_{m,n}(w)$ and u such that $\bar{w}u\bar{w}'$ is a word in Σ we define $\mathcal{W}^*(u)$ to be the set of Σ words $w_1 u_1 w'_1$ such that u and u_1 are pairwise related and

$w_1, w_1' \in \mathcal{W}_{m,n}(w)$. Therefore, by the size of K we know that words in $\mathcal{W}^*(u)$ are pairwise related. Fix a u such that $\mathcal{W}^*(u)$ is nonempty.

Claim 7.6. *$\mathcal{W}^*(u)$ is a maximal family of related words.*

Proof. Suppose x is a word pairwise related to a word $w^* = w_1 u_1 w_1' \in W^*(u)$. Then $x = x_1 y_1 x_1'$ where $|x_1| = |x_1'| = |n - m|$ and $|y_1| = |u|$. From the size of K we know x_1 and x_1' are pairwise related to w_1 and w_1', respectively. Hence, $x_1, x_2 \in W_{m,n}(w)$. This implies that $x \in W^*(u)$. □

So $\deg(\mathcal{W}^*(u), 0) \geq d$ and $\deg(\mathcal{W}^*(u), 0) \leq \deg(\mathcal{W}_{m,n}, 0) = d$. Therefore, $\deg(W^*(u), 0) = d$.

Claim 7.7. *Let $\{i_1, ..., i_d\} = \{\bar{w}_0 \,|\bar{w} \in \mathcal{W}_{m.n}(w)\}$ and $w^* \in \mathcal{W}^*(u)$. Then there is a permutation τ of $\{i_i, ..., i_d\}$ such that for $1 \leq j \leq d$ there exists a unique word $v = v(i_j)$ such that $i_j v \tau(i_j) = w^*[0, |u| + n + 1]$.*

Proof. We first show that v is unique. Suppose for some $j \in \{1, ...d\}$ there exists words $i_j v \tau(i_j)$ and $i_j v' \tau(i_j)$. Then there exist points $t, t' \in \Sigma$ such that

- $t \neq t'$,
- $t[0, |u| + n + 1] = i_j v \tau(i_j)$,
- $t'[0, |u| + n + 1] = i_j v' \tau(i_j)$, and
- $t_i = t_i'$ for $i < 0$ and $i > |v| + 2$.

Then $\phi(t) = \phi(t')$ and $t \in W^u(t') \cap W^s(t')$. From Lemma 2.1 in [13] we know $t = t'$, a contradiction.

We now show that v exists. For $1 \leq j \leq d$ there exists some $w^* \in W^*(u)$ such that $w_0^* = i_j$. Otherwise, as stated above $\deg(\mathcal{W}^*(u), 0) < d$, a contradiction. Similarly, there exists some $w^* \in W^*(u)$ such that $w^*_{|u|+1} = i_j$.

Suppose for some $1 \leq j \leq d$ there exists $k_1, k_2 \in \{1, ...d\}$ and words v' and v'' such that $i_j v' i_{k_1} = w_1^*[0, |u|+n+1]$ and $i_j v'' i_{k_2} = w_2^*[0, |u|+n+1]$ for some $w_1^*, w_2^* \in \mathcal{W}^*(u)$.

Define $\mathcal{X}$ to be the collection of all words $w^*[1, |u| + n + 1] = v i_s$ for some $w^* \in \mathcal{W}^*(u)$. Since $n \geq K$ we know that each of the words in $\mathcal{X}$ are pairwise related. Let $\mathcal{X}^k$ be the collection of allowed concatenations of words $x_1 \cdots x_k$ where each $x_j \in \mathcal{X}$. Then any two elements in $\mathcal{X}^k$ are pairwise related for each $k \in \mathbb{N}$. We claim that $\#(\mathcal{X}^k) < \#(\mathcal{X}^{k+1})$. Let $x^{(k)} \in \mathcal{X}^k$ and $x^{(k)}$ end in i_s. Then there exists some word $w^* \in \mathcal{W}^*(u)$ where $w^*[0, |u| + n + 1] = i_s v i_t$ for some $t \in \{1, ..., d\}$. Therefore, we know that each word $x^{(k)} \in \mathcal{X}^k$ has an extension to a word $x^{(k+1)} \in \mathcal{X}^{k+1}$ and $\#(\mathcal{X}^k) \leq \#(\mathcal{X}^{k+1})$. Now let $x^{(k)} \in \mathcal{X}^k$ such that $x^{(k)}$ ends in i_j. Then

$$x^{(k)} v' i_{k_1}, x^{(k)} v'' i_{k_2} \in \mathcal{X}^{k+1}$$

and are distinct. Hence, $\#(\mathcal{X}^k) < \#(\mathcal{X}^{k+1})$.

This implies that $\mathcal{X}^{D+1}$ has more than D elements all of which can be extended to points in Σ mapping to the same point in Y, a contradiction. $\square$

Claim 7.8. *Choose $t \in \Sigma$ such that a word in $\mathcal{W}_{m,n}(w)$ occurs infinitely often in the past and future. Then $\#(\phi^{-1}(\phi(t))) = d$ and there does not exist a point $t' \in \Sigma$ such that $t' \in W^s(t) \cap \phi^{-1}(\phi(t))$ and $t' \neq t$.*

Proof Let $\bar{w}$ be the word in $\mathcal{W}_{m,n}(w)$ occurring infinitely often in t. We know that there exists $N, M \in \mathbb{N}$ such that $t[-N, M] = \bar{w}u\bar{w}$ for some word u. From the previous claim there exist d words related to $t[-N, M]$. As N and M can be chosen arbitrarily large we know that $\phi^{-1}(\phi(t)) = d$.

Now suppose that $t' \in W^s(t) \cap \phi^{-1}(\phi(t))$. We may suppose that $t'_i = t_i$ for all $i \geq 0$ and $t[0, m+n] = \bar{w}$. Then there exists $N \in \mathbb{N}$ such that $t[-N, m+n] = \bar{w}u\bar{w}$ for some word u and $t'[-N, m+n] = w'u'\bar{w}$ where $w' \in \mathcal{W}_{m,n}(w)$. From the previous claim we know that $t'[-N+m, m+n] = t[-N+n, m+n]$. Since N can be made arbitrarily large we have $t' = t$. $\square$

In particular, the above claim shows that

$$d \geq \min_{y \in Y} \#\phi^{-1}(y) = D$$

so $d = D$. Furthermore, if $t' \in \phi^{-1}(\phi(t))$, then t and t' are pairwise related, t' has words of $\mathcal{W}_{m,n}$ occurring infinitely often in the past and future, and the words occur in the same location as those of t.

The next claim says that if there is a finite-to-one factor map from a transitive SFT to a finitely presented system and two points on the same stable set map to the same point, then for some iterate the points are on the unstable boundaries of their respective rectangles. This will be a helpful characterization in the proof of Theorem 1.4 since θ was defined to separate such points. This will imply that π_2 is s-resolving.

Claim 7.9. *Suppose Σ is a transitive one-step SFT, Y is a transitive Smale spaces, and $\phi : \Sigma \to Y$ is a finite-to-one factor map. If $t, t' \in \Sigma$ such that $\phi(t) = \phi(t') = y$ and $t \in W^s(t')$, then for some $j \in \mathbb{Z}$ the point*

$$\phi(\sigma^j(t')) \in \partial_u^{t_j} R_{t'_j} \cap \partial_u^{t'_j} R_{t_j}.$$

Proof. We want to show that

$$\phi(t) = \phi(t') \in \partial_u^{t_0} R_{t'_0}$$

so there exists a sequence

$$y_i \in (W^s_\epsilon(\phi(t')) \cap R_{t_0}) - R_{t'_0}$$

converging to $\phi(t')$. The argument that

$$\phi(t) = \phi(t') \in \partial_u^{t'_0} R_{t_0}$$

will be similar and left to the reader.

We may suppose that $t_i = t'_i$ for all $i \geq 1$ and $t_0 \neq t'_0$. Let $\mathcal{W}_{m,n}(w)$ have degree d. Since Σ is transitive we know that for each $M \in \mathbb{N}$ there exists a $t^M \in \Sigma$ such that $t_i^M = t_i$ for all $-M \leq i \leq M$ and words from $\mathcal{W}_{m,n}(w)$ occur infinitely often in the past and in the future for t^M.

Let $s^M = [t, t^M]$. Then $\phi(s^M) \in W^s(\phi(t), R_{t_0})$. We will show that

$$\phi(s^M) \notin W^s(\phi(t'), R_{t'_0}).$$

As $M \to \infty$ we know that $\phi(s^M) \to \phi(t)$. This implies that $\phi(t) \in \partial_u^{t_0} R_{t'_0}$.

Suppose that

$$\phi(s^M) \in W^s(\phi(t'), R_{t'_0}).$$

Then there exists $s \in \Sigma$ such that

- $s_0 = t'_0$,
- $\phi(s) = \phi(s^M)$, and
- $\phi(s) \in W^s(\phi(t'), R_{t'_0})$.

Let $x[0, \infty)$ be the infinite sequence such that

$$x_0 = t'_0, \text{ and } x_i = t_i^M \text{ for all } i \geq 1.$$

Let $s' \in \Sigma$ such that $s'[0, \infty) = x[0, \infty)$ and $s'_i = s_i$ for all $i \leq 0$. We know that for all $i \leq 0$ we have

$$s'_i \sim_\phi s_i^M = t_i^M,$$

and $s'_i = t_i^M$ for all $i \geq 1$. Furthermore,

$$(s'_i, t_i^M) \to (s'_{i+1}, t_{i+1}^M)$$

is an allowed transition in E_ϕ for all $i \in \mathbb{Z}$. Then

$$\phi(s') = \phi(t^M) \text{ and } s' \in W^s(t^M),$$

a contradiction. Hence, $\phi(s^M) \notin W^s(\phi(t'), R_{t'_0})$. □

Before proceeding to the proof of Lemma 7.4 we need to prove some additional properties concerning pre-images of periodic points and transitive points under π_+^0.

Claim 7.10. *Let Σ be a transitive 1-step SFT and $\phi : \Sigma \to Y$ is a finite-to-one factor map onto a finitely presented system (Y, f). If $p \in \mathrm{Per}(Y, f)$ and there exist $s, s' \in \Sigma$ such that $\phi(s) = \phi(s') = p$ and $s_0 = s'_0$, then $s = s'$.*

The proof of Claim 7.10 is similar to the proof of Lemma 3.1 and is left to the reader.

Claim 7.11. *Let Σ be a transitive 1-step SFT, (Y, f) be finitely presented, and $\phi : \Sigma \to Y$ be finite-to-one. Let D be the minimum number of pre-images under ϕ, the set $W \subset Y$ be open and dense such that each periodic point in W has D preimages under ϕ and every point in W is contained in the interior of a rectangle. If $p \in W$ is a periodic point, then $\#(\pi_+^0)^{-1}(p) = D$.*

Proof. From Lemma 3.2 we know that the set W exists. We know that for any rectangle containing p that p is contained in the interior of the rectangle. Let $\omega \in \Sigma_+$ such that $\pi_+^0(\omega) = p$ and let $\omega_0 = (v, i)$. Then there exists some $s \in \phi^{-1}(p)$ such that $s_0 = i$. Let $\omega' \in \Sigma_+$ be canonically associated with s and $\omega'_0 = (v', i)$. Fix a sequence $s_k \in \Sigma$ such that the canonically associated sequence $\omega_k \in \Sigma_+$ converges to ω. We may assume the s_k are convergent. From the previous claim we know that the sequence s_k converges to s.

Suppose there exists a $j \in v$ such that $j \notin v'$. Then there exists a sequence $t_k \in \Sigma$ such that $\phi(t_k) \in W^u(\phi(s_k), R_i) \cap R_j$. We may assume that t_k converges to some point t. Hence, $\phi(t) \in W^u(\phi(s), R_i) \cap R_j$ and $j \in v'$, a contradiction.

Suppose there exists a $j \in v'$ such that $j \notin v$. Let $y \in W^u(p, R_i) \cap R_j$ and $N \in \mathbb{N}$ be the period of p. Then $f^{-mN}(y) \in \mathrm{int}(R_i)$ for some $m \in \mathbb{N}$ since $p \in \mathrm{int}(R_i)$. Fix such an $m \in \mathbb{N}$. Then there exists a $K \in \mathbb{N}$ such that $(s_k)_{-lN} = i$ for all $0 \leq l \leq m$ where $k \geq K$. Such a K exists since $\sigma^{-lN}(s_k) \to s$ as $k \to \infty$ for all l. For $k \geq K$ let

$$y_k = [f^{-mN}(y), \pi(\sigma^{-mN}(s_k))] \in R_i.$$

Since $f^{-mN}(y) \in R_i \cap f^{-mN}(R_j)$ we know from the Markov property of the rectangles that $f^{mN}(y_k) \in R_j$ for all $k \geq K$. Furthermore, we know that

$$f^{lN}(y_k) = [f^{(l-m)N}(y), \pi(\sigma^{(l-m)N}(s_k)]$$

for all $0 \leq l \leq m$ since this point is unique. Then

$$f^{mN}(y_k) \in W^u(\pi(s_k), R_i) \cap R_j$$

for all $k \geq K$. This implies that $j \in v'$, a contradiction. Therefore, $v = v'$ and $\omega_0 = \omega'_0$. Since p is periodic we see by iterating p that $\omega = \omega'$. □

Remark 7.12. *The previous claim shows that if Σ is a transitive 1-step SFT, (Y, f) is finitely presented, $\phi : \Sigma \to Y$ is finite-to-one, and D is the minimum number of pre-images under ϕ, then*

1. *D is the minimum number of pre-images under π_+^0, and*
2. *there exists an open and dense set W of Y such that if $p \in \mathrm{Per}(Y, f) \cap W$ and $\omega, \omega' \in (\pi_+^0)^{-1}(p)$, then $D = \#\phi^{-1}(p)$ and*

> *$\theta(\omega) = \theta(\omega')$. This follows since ω and ω' are canonically associated with points s and s' in Σ, respectively, where $\phi(s) = \phi(s')$ and we know that such points map to the same point under θ.*

.

Claim 7.13. *If Σ is a transitive 1-step SFT, (Y, f) is finitely presented, $\phi : \Sigma \to Y$ is finite-to-one, D is the minimum number of pre-images under ϕ, and $y \in Y$ is transitive, then $\#(\phi)^{-1}(y) = \#(\pi_+^0)^{-1}(y) = D$ and $\#(\pi_+)^{-1}(y) = 1$.*

Proof. Let W be as in the statement of Claim 7.11. We know that

$$\min_{y\in Y} \#(\phi)^{-1}(y) = D = \min_{y\in Y} \#(\pi_+^0)^{-1}(y)$$

and for any transitive point $y \in Y$ that $\#(\phi^{-1}(y)) = D$ and $\#(\pi_+^0)(y) = D$. Then the preimages of y for π_+^0 are the canonically associated points from the preimages of y for ϕ. Since all the canonically associated points map to the same point under θ we know that $\#(\pi_+)^{-1}(y) = 1$. $\square$

Claim 7.14. *Suppose Σ is a transitive one-step SFT, Y is a transitive Smale spaces, and $\phi : \Sigma \to Y$ is a finite-to-one factor map. Let $t, t' \in \Sigma$ such that $\phi(t) = \phi(t') = y$, $t \in W^s(t')$, $t_0 \neq t'_0$, and $t_i = t'_i$ for all $i \geq 1$. If s is a transitive point of Σ sufficiently close to t and $\omega \in (\pi_+^0)^{-1}(\phi(s))$, then $t_0 \in V_\delta^u(\omega)$ and $t'_0 \notin V_\delta^u(\omega)$.*

Proof. Since s is transitive we know that words from $\mathcal{W}_{m,n}(w)$ occur infinitely often in the past and in the future for s. From the proof of Claim 7.9 we know that

$$\phi(t) = \phi(t') \in \partial_u^{t_0} R_{t'_0} \cap \partial_u^{t'_0} R_{t_0}.$$

Furthermore, from the proof of Claim 7.9 if $s_i = t_i$ for all $-M \leq i \leq M$ and M is sufficiently large, then

$$\phi([t, s]) \notin W^s(\phi(t'), R_{t'_0}).$$

Hence, $\phi([t, s]) \notin R_{t'_o}$.

If there exists a point $z \in W^u(\phi(s), R_{t_0}) \cap R_{t'_0}$, then

$$[z, \phi(t)] = [\phi(s), \phi(t)] = \phi[s, t] \in R_{t'_0},$$

a contradiction. Hence,

$$W^u(\phi(s), R_{t_0}) \cap R_{t'_0} = \emptyset.$$

If $\omega \in (\pi_+^0)^{-1}(\phi(s))$, then ω is canonically associated with a preimage of $\phi(s)$ under ϕ^{-1}. Therefore, $V_\delta^u(\omega)$ contains t_0 and does not contain

t_0'. □

Proof of Lemma 7.4 We know that $\bar{Y}$ is the unique irreducible component in $\theta(\Sigma_+)$. This implies that the fiber product of Σ and $\bar{Y}$ is transitive. Hence, $G = G_0$. Suppose there exists points $(t, y), (t', y) \in G$ where $t \in W^s(t')$. We may assume that $t_0 \neq t_0'$ and $t_i = t_i'$ for all $i \geq 1$.

Let $(s, z) \in G$ be transitive. By choosing iterates of (s, z) sufficiently close to (t, y) and (t', y) we then know from the characterization of transitive points in Remark 7.12 that $t_0, t_0' \in V_\delta^u(\omega)$ for all $\omega \in \Sigma_+$ such that $\theta(\omega) = y$.

However, from Claim 7.14 we know that for all iterates of (s, z) sufficiently close to (t, y) that $t_0' \notin V_\delta^u(\omega)$ where ω is contained in the preimage of the iterate of $\tilde{\pi}\pi(s)$ under π_+^0, a contradiction. Therefore, π_2 is s-resolving. □

References

[1] N. Aoki and K. Hiraide. *Topological Theory of Dynamical Systems, Recent Advances*. North-Holland, 1994.

[2] M. Boyle. Factoring factor maps. *J. London Math. Soc. (2)*, 57(2):491–502, 1998.

[3] M. Boyle. Putnam's resoving maps in dimension zero. *Ergod. Th. and Dyn. Sys.*, 25(5):1485–1502, 2005. preprint.

[4] M. Boyle, B. Kitchens, and B. Marcus. A note on minimal covers for sofic systems. *Proc. Amer. Math. Soc.*, 95(3):403–411, November 1985.

[5] M. Brin and G. Stuck. *Introduction to Dynamical Systems*. Cambridge University Press, 2002.

[6] R. Fischer. Graphs and symbolic dynamics. *Colloq. Math. Soc. János Bólyai: Topics in Information Theory*, 16:229–243, 1975.

[7] R. Fischer. Sofic systems and graphs. *Monats. Math.*, 80:179–186, 1975.

[8] T. Fisher. Hyperbolic sets that are not locally maximal. *Ergod. Th. Dynamic. Systems*, 26(5):1491–1509, 2006. *Ergod. Th. Dynamic. Systems*.

[9] D. Fried. Finitely presented dynamical systems. *Ergod. Th. and Dyn. Sys.*, 7:489–507, 1987.

[10] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, 1995.

[11] J. Ombach. Equivalent conditions for hyperbolic coordinates. *Topology Appl.*, 23(1):87–90, 1986.

[12] I. Putnam. Functoriality of the C^*-algebras associated with hyperbolic dynamical systems. *J. London Math. Soc. (2)*, 62:873–884, 2000.

[13] I. Putnam. Lifting factor maps to resolving maps. *Isreal J. Math.*, 146:253–280, 2005.

[14] D. Ruelle. *Thermodynamic Formalism*. Addison Wesley, Reading, 1978.

[15] S. Williams. A sofic system with infinitely many minimal covers. *Proc. Amer. Math. Soc.*, 98(3):503–506, 1986.

[16] S. Williams. Covers of non-almost-finite type sofic systems. *Proc. Amer. Math. Soc.*, 104(1):245–252, 1988.

Department of Mathematics Brigham Young University, Provo, UT 84602

E-mail address: tfisher@math.byu.edu